

Generalizing Geometry—Algebroids and Sigma Models

Alexei Kotov¹ and Thomas Strobl²

¹ *Universite du Luxembourg, Laboratoire de Mathematiques,
Campus Limpertsberg, 162A, Avenue de la Faiencerie, L-1511 Luxembourg
e-mail address: alexei.kotov@uni.lu*

² *Univ. de Lyon, Univ. Lyon1, CNRS UMR 5208, Institut Camille Jordan,
43 Boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France
email address: strobl @ math.univ-lyon1.fr*

Abstract

In this contribution we review some of the interplay between sigma models in theoretical physics and novel geometrical structures such as Lie (n-)algebroids. The first part of the article contains the mathematical background, the definition of various algebroids as well as of Dirac structures, a joint generalization of Poisson, presymplectic, but also complex structures. Proofs are given in detail. The second part deals with sigma models. Topological ones, in particular the AKSZ and the Dirac sigma models, as generalizations of the Poisson sigma models to higher dimensions and to Dirac structures, respectively, but also physical ones, that reduce to standard Yang Mills theories for the “flat” choice of a Lie algebra: Lie algebroid Yang Mills theories and possible action functionals for nonabelian gerbes and general higher gauge theories. Characteristic classes associated to Dirac structures and to higher principal bundles are also mentioned.

Contents

1	Introduction	2
2	Lie and Courant algebroids	7
3	Dirac structures	16
4	Generalized complex structures	21
5	Algebroids as Q-manifolds	24
6	Sigma models in the AKSZ-scheme	31
7	Sigma models related to Dirac structures	38
8	Yang-Mills type sigma models	42
	References	50

1 Introduction

In this article we want to summarize some ideas in the overlap of differential geometry and mathematical physics. In particular we focus on the interplay of so-called sigma models with geometrical structures being related to algebroids in one way or another. Several traditional geometrical notions received various kinds of generalizations in recent years. Some of them give rise to novel sigma models, while also some sigma models bring attention to possible new geometries.

Sigma models are action functionals (variational problems) where the underlying space of fields (maps) has a target space equipped with some geometry; in the most standard case one regards maps \mathcal{X} from one given Riemannian manifold (Σ, h) to another one (M, g) and considers the functional

$$S[\mathcal{X}] = \frac{1}{2} \int_{\Sigma} \|\mathrm{d}\mathcal{X}\|^2, \quad (1.1)$$

where $\|\alpha\|^2 = (\mathcal{X}^*g)(\alpha \wedge * \alpha)$ for any $\alpha \in \Omega^p(\Sigma, \mathcal{X}^*TM)$ with $*$ denoting the h -induced Hodge duality operation on Σ ,¹ the critical points of which are precisely the maps \mathcal{X} which are harmonic. For the special case that M is just \mathbb{R}^n equipped with the standard flat metric this functional reduces to

$$S[\phi^i] = \frac{1}{2} \int_{\Sigma} \mathrm{d}\phi^i \wedge * \mathrm{d}\phi^i, \quad (1.2)$$

the action of n “free scalar fields” on Σ (here ϕ^i denotes the function on Σ obtained by restricting \mathcal{X} to the i -th coordinate in \mathbb{R}^n —it is understood that the index i on the l.h.s. is an “abstract index”, i.e. S depends on all the scalar fields, ϕ^1 to ϕ^n).

Another example of a sigma model is the Poisson sigma model [36, 21], where the source manifold Σ is necessarily two-dimensional and the target manifold M carries a Poisson structure instead of a Riemannian one. In fact, one considers a functional on the space of vector bundle morphisms from $T\Sigma$ to T^*M in this case. It is “topological”, which we want to interpret as saying that the space of classical solutions (stationary points of the functional) modulo gauge transformations (invariances or symmetries of the functional) is finite dimensional—besides the fact that it does not depend on geometrical structures of the source like a Riemannian metric h (which in this case is even absent in the definition of the functional). The tangent bundle of any manifold as well as the cotangent bundle of a Poisson manifold give rise to what are called Lie algebroids (whose definition is properly recalled in the body of the paper below—cf. in particular Examples 3.2 and 2.3 below) and the above functional stationarizes precisely on the morphisms of these Lie algebroid structures [4].

But this also works the other way around: given a functional of such a form defined by an a priori arbitrary bivector field Π (and, in some particular extension of the functional,

¹In local coordinates σ^μ on Σ and x^i on M this expression reads more explicitly as $\alpha_{i\mu_1 \dots \mu_p} \alpha^{i\mu_1 \dots \mu_p} \mathrm{dvol}_{\Sigma}$ where indices μ_i and i are raised and lowered by means of the metric h and (the pullback by \mathcal{X} of) g , respectively, and $\mathrm{dvol}_{\Sigma} = \sqrt{\det(h)} \mathrm{d}^d \sigma$, d denoting the dimension of Σ . In this article we use the Einstein sum convention, i.e. a sum over repeated indices is always understood.

also a closed 3-form H) the respective functional becomes topological, *iff* [35, 23] Π defines a (for non-zero H twisted) Poisson structure. In fact, twisted Poisson structures (c.f. also [34]), i.e. bivectors Π together with closed 3-forms H defined over a manifold M satisfying

$$[\Pi, \Pi] = (\Pi^\natural)^{\otimes 3} H \quad (1.3)$$

where [...] denotes the Schouten-Nijenhuis bracket of multi-vector fields and $\Pi^\natural: T^*M \rightarrow TM$ is the natural operator induced by the bivector Π as follows: $\Pi^\natural(\alpha) = \Pi(\alpha, \cdot)$, were even *found* first in such a manner [23, 30]. This is typical for the interplay of geometrical notions and sigma models: the former are needed to define the latter, but sigma models sometimes also give indications about (focus on) particularly interesting geometrical notions. In this example, in the space of bivectors on a manifold the ones which are Poisson are singled out by the sigma model, or in the space of pairs (Π, H) those satisfying eq. (1.3) (which can be seen to define a particular Dirac structure [34]—we introduce to Dirac structures in the main text in detail). Another example for such an interplay are supersymmetric sigma models and bihermitian geometry [14]. The latter geometry received renewed and revived interest recently by its elegant reformulation in terms of so-called generalized complex structures [20].

What is, on the other hand, meant more specifically by generalizing traditional geometrical notions? In fact, also the generalized notions can usually be expressed in terms of ordinary differential geometrical ones, in which case it just boils down to a different way of thinking about them. In any such case one has usually some particular kind of a so-called *algebroid* in the game.

There are several kinds of an algebroid considered in the literature. All of them have the following data in common, which we thus want to use as a definition of the general term:

Definition 1.1 *We call an algebroid $E = (E, \rho, [\cdot, \cdot])$, a vector bundle $E \rightarrow M$ together with a homomorphism of vector bundles $\rho: E \rightarrow TM$, called the anchor of E , and a product or bracket on the sections of E satisfying the Leibniz rule ($\psi, \psi' \in \Gamma(E)$, $f \in C^\infty(M)$):*

$$[\psi, f\psi'] = f[\psi, \psi'] + (\rho(\psi)f) \psi'. \quad (1.4)$$

Note that the map on sections induced by ρ is denoted by the same letter conventionally. Depending on further conditions placed on the bracket (like its symmetry properties or its Jacobiator) and further structures defined on E , one has different kinds of algebroids: Lie algebroids, Courant algebroids, strongly homotopy Lie (or L_∞ -) algebroids, and so on. Lie algebroids are obtained e.g. by requiring in addition that the bracket is a Lie bracket, i.e. antisymmetric and with vanishing Jacobiator. The definition of a Courant algebroid requires a fiber metric on E controlling the symmetric part of the bracket as well as its Jacobiator. We will come back to all these various kinds of algebroids in more detail in the text below.

The philosophy now is that we can do differential geometry by replacing TM or, more generally, also the tensor bundle $\tau_q^p(M) = TM^{\otimes p} \otimes T^*M^{\otimes q}$ by E and $E^{\otimes p} \otimes E^{*\otimes q}$, respectively. Let us call a section t of $\oplus_{p,q} (E^{\otimes p} \otimes E^{*\otimes q})$ an E -tensor field. The Leibniz property permits us to define a (Lie) “derivative” of t along any section ψ of E : Indeed, set ${}^E L_\psi(f) := \rho(\psi)f$, ${}^E L_\psi(\psi') := [\psi, \psi']$, and extend this to powers of E by the Leibniz property w.r.t. tensor multiplication and to E^* by means of compatibility with contraction: ${}^E L_\psi(\langle \psi', \omega \rangle) = \langle {}^E L_\psi(\psi'), \omega \rangle + \langle \psi', {}^E L_\psi(\omega) \rangle$, defining the Lie derivative for any $\omega \in \Gamma(E^*)$. This implies that given an algebroid E (not necessarily Lie), i.e. data that at least include those of the definition above, we can define a “Lie derivative” of any E -tensor along sections of E . By construction, this generalizes the notion of an ordinary Lie derivative of ordinary tensor fields: indeed, the usual formulas are reproduced in the case where the algebroid is chosen to be a so-called standard Lie algebroid, i.e. where E is the tangent bundle of the base manifold M , the anchor ρ is the identity map and the bracket is the standard Lie bracket of vector fields. Such type of geometrical notions have properties in common with their prototypes in traditional geometry (like, in this case e.g., by construction, the Leibniz property of the generalized Lie derivatives), but they in general also have pronounced differences, possibly depending on the type of algebroid, i.e. on the additional structures. In the above example one can ask e.g. if the commutator of such Lie derivatives is the Lie derivative of the bracket of the underlying sections. In general this will not always be the case. However, for an important subclass, containing Lie and Courant algebroids, it will (cf. Lemma 2.6 below).

Another important example of such generalized structures is “generalized geometry” or *generalized complex geometry* in the sense of Hitchin (cf. also the contribution of N. Hitchin to this volume). It is a particular case of the above viewpoint where E is taken to be $TM \oplus T^*M$ equipped with projection to the first factor as anchor as well as the so-called Courant or Dorfman bracket (in fact, slightly more general and conceptually preferable, one takes E to be what is called an exact Courant algebroid—cf. definition 2.14 below). Now, by definition a generalized complex structure is what usually would be a complex structure, just replacing the standard Lie algebroid TM by an exact Courant algebroid. In particular, it is an endomorphism of $E \cong TM \oplus T^*M$ squaring to minus one and satisfying an integrability condition using the bracket on E (cf., e.g., Prop. 4.3 below). It is not difficult to see that this notion generalizes simultaneously ordinary complex structures as well as symplectic ones. In fact, the situation is closely related to (real) Dirac structures (particular Lie subalgebroids of exact Courant algebroids, cf. definition 3.1 below), mentioned already previously above: these generalize simultaneously Poisson and presymplectic structures on manifolds. In fact, generalized complex structures did not only find at least part of their inspiration from real Dirac structures, but they can be even defined equivalently as imaginary Dirac structures—which is the perspective we want to emphasize in the present note.²

An elegant and extremely useful viewpoint on some algebroids arises within the language of differential graded manifolds, sometimes also called Q -manifolds (Q denoting a homological degree one vector field on the graded manifold, i.e. its differential). We de-

²This point of view was maybe less known at the time when we this note was started, while in the mean time it has received some attention also elsewhere.

vote a section to explaining this relation, after having introduced the reader to the above mentioned notions of algebroids, Dirac structures, and generalized complex structures, in the three sections to follow. Together these four sections provide our exposition on algebroids and this kind of generalized geometry.

Some of the following sections then deal with the respective sigma models: Given a Q-manifold with a compatible graded symplectic structure on it, one can always associate a topological sigma model to it [1]. We review this construction in some detail and specialize it to lowest dimensions (of the source manifold Σ), reproducing topological models corresponding to Poisson manifolds ($\dim \Sigma = 2$, this is the above mentioned Poisson sigma model), to Courant algebroids ($\dim \Sigma = 3$, such models were considered in [31]). Some space is devoted to describing these models, somewhat complementary to what is found in the literature, since they can be used to introduce part of the formalism that is needed for the last section to this contribution.

There are also topological models that, at least up to now, have not yet been related to the AKSZ formalism and corresponding e.g. to Dirac structures. We recall these models, called Dirac sigma models [25] and generalizing the Poisson sigma models essentially such as (real) Dirac structures generalize Poisson manifolds, in a separate section. These as well as the AKSZ models share the property that they are topological and that the solutions to their field equations generalize (only) *flat* connections to the algebroid setting.

The final section is devoted to sigma models in arbitrary spacetime dimension (dimension of Σ which then is taken to be pseudo-Riemannian) with a relation to algebroids but which are nontopological and which generalize connections that are not necessarily flat but instead satisfy the Yang-Mills field equations.³ This deserves some further motivating explanation already in the introduction:

Consider replacing in eq. (1.2) the functions on Σ by 1-forms A^a , $a = 1, \dots, r$, yielding

$$S[A^a] = \frac{1}{2} \int_{\Sigma} dA^a \wedge *dA^a. \quad (1.5)$$

For $r = 1$, i.e. there is just one 1-form field A , this is a famous action functional, describing the electromagnetic interactions (the electric and the magnetic fields can be identified with the components of the “field strength” 2-form dA). Having several such 1-forms in the game, $r > 1$, one obtains the functional describing r free (i.e. mutually independent) 1-form fields⁴. The most standard way of making scalar fields interact is to add some “potential term” to the functional (1.2), i.e. to add the integral over $V(\phi^i)$ multiplied by the h -induced volume form on Σ (where V is some appropriately smooth function on

³Being nontopological is required for most physical applications so as to host the degrees of freedom necessary to describe realistic interactions. — There are also another type of (nontopological) sigma models than those explained in the last section that are related to algebroids and in particular generalized and bihermitian geometry. These are supersymmetric two-dimensional sigma models, i.e. *string theories*. Although also highly interesting, we will not touch this issue here but refer e.g. to the review article [42] and references therein.

⁴In the physics language they are often called “vector fields” (as opposed to “scalar fields” used in (1.2)). We avoid this somewhat misleading/ambiguous nomenclature, but we will, however, from time to time refer to them as “gauge fields”, despite the fact that in the mathematical setting they correspond to connections in a principal G -bundle (here trivialized with $G = U(1)^r$).

\mathbb{R}^n , mostly even only a low degree polynomial so as to not spoil “renormalizability”).⁵ Turning (1.5) into an interacting theory (without introducing further fields and not spoiling its gauge invariance, at most “deforming” the latter one appropriately) is not so simple. In fact, the result is rather restricted (cf., e.g., uniqueness theorems in the context of the deformation theory of gauge theories [3]) and one is lead to only replace dA^a by the expression

$$F = dA + \frac{1}{2}[A \wedge A] \quad (1.6)$$

for the curvature of the Lie algebra valued connection 1-form A of a (trivialized) principal bundle for some r -dimensional, quadratic Lie group G . “Quadratic” means that its Lie algebra \mathfrak{g} admits an ad-invariant inner product κ which, when starting by deforming (1.5), needs to have definite signature such that a sum over the index a results from a term of the form

$$S_{YM}[A] = \int_{\Sigma} \kappa(F \wedge *F) \quad (1.7)$$

after choosing a κ -orthonormal basis in \mathfrak{g} . The resulting theory is called a Yang-Mills (gauge) theory and was found to govern all the strong, the weak, and the electromagnetic interactions.

If one considers an algebroid E as defined above over a zero dimensional base manifold M , M degenerating to a point, one is left only with a vector space. For E being a Lie algebroid this vector space becomes a Lie algebra, for E being a Courant algebroid it becomes a quadratic Lie algebra. In fact, this is a second, algebraic part that is incorporated in algebroids: general Lie algebroids can be thought of as a common generalization of the important notions of, on the one hand, a Lie algebra \mathfrak{g} and, on the other hand, standard geometry (i.e. geometry defined for TM or $\tau_q^p(M)$), as we partially explained already above. (A similar statement is true for general Courant algebroids, TM then being replaced by the “standard Courant algebroid” $TM \oplus T^*M$ and \mathfrak{g} by a quadratic Lie algebra together with its invariant scalar product). From this perspective it is thus tempting to consider the question if one can define e.g. a theory of principal bundles with connections in such a way that the structural Lie algebra is replaced by (or better generalized to) appropriately specified structural algebroids. Likewise, from the more physical side, can one generalize a functional such as the Yang-Mills functional (1.7) to a kind of sigma model, replacing the in some sense flat Lie algebra \mathfrak{g} by nontrivial geometry described via some appropriate Lie or Courant algebroid? These questions will be addressed and, at least part of them, answered to the positive in the final section to this article.

Between the part on sigma models and the one on algebroids it would have been nice to also include a section on current algebras (cf. [2] as a first step), as another link between the two; lack of spacetime, however, made us decide to drop this part in the present contribution.

⁵An alternative way of having scalar fields interact is coupling them to 1-form gauge fields so that they start being correlated (i.e. interacting) via these 1-form fields. In fact, both ways of interactions are realized in the standard model of elementary particle physics, where the gauge fields describe interaction particles like the photon and the scalar fields describe “matter”, essentially like electrons (or the–not yet discovered–Higgs particles).

2 Lie and Courant algebroids

In the present section, we recall the notions of Lie and Courant algebroids in a rigorous manner and study some of their properties. We also briefly introduce some of their higher analogues, like Lie 2-algebroids and vector bundle twisted Courant algebroids. In a later section we will provide another, alternative viewpoint of all these objects by means of graded manifolds, which permits an elegant and concise reformulation.

Definition 2.1 *A Loday algebroid is an algebroid $(E, \rho, [\cdot, \cdot])$ as defined by Definition 1.1 where the bracket defines a Loday algebra on $\Gamma(E)$, i.e. it satisfies the Loday (or left-Leibniz) property,*

$$[\psi_1, [\psi_2, \psi_3]] = [[\psi_1, \psi_2], \psi_3] + [\psi_2, [\psi_1, \psi_3]]. \quad (2.1)$$

An almost Lie algebroid is an algebroid E where the bracket is antisymmetric. E becomes a Lie algebroid, if $[\cdot, \cdot]$ defines a Lie algebra structure on $\Gamma(E)$, i.e. if E is simultaneously Loday and almost Lie.

Here we adapted to the nomenclature of Kosmann-Schwarzbach, who prefers to use the name of Loday in the context of (2.1) so as to reserve the terminus Leibniz for compatibility w.r.t. multiplication of sections by functions, (1.4), for which “Leibniz rule” has become standard.

Example 2.2 *Obviously, $(TM, \rho = \text{Id})$ together with the Lie bracket of vector fields is a Lie algebroid; it is called the standard Lie algebroid.*

If M is a point, on the other hand, a Lie algebroid reduces to a Lie algebra. More generally, if the anchor of a Lie algebroid map vanishes (i.e. maps E to the image of the zero section in TM), E is a bundle of Lie algebras; in general not a Lie algebra bundle, since Lie algebras of different fibers need not be isomorphic.

Example 2.3 *A less trivial Lie algebroid is the cotangent bundle T^*M of a Poisson manifold.⁶ The anchor is provided by contraction with the Poisson bivector field Π and the bracket of exact 1-forms $[df, dg] := d\{f, g\}$ is extended to all 1-forms by means of the Leibniz rule (1.4).*

Example 2.4 *An example of a Loday algebroid with a non-antisymmetric bracket is the following one: $E = TM \oplus T^*M$ with ρ being projection to the first factor and the bracket being given by*

$$[\xi + \alpha, \xi' + \alpha'] = [\xi, \xi'] + L_\xi \alpha' - \iota_{\xi'} d\alpha; \quad (2.2)$$

here ξ, ξ' and α, α' denote vector fields and 1-forms on M , respectively, and the bracket on the r.h.s. is the usual Lie bracket of vector fields. This is the so-called Dorfmann bracket.

⁶By definition, one obtains a Poisson structure on a smooth manifold M , if the space of functions is equipped with a Lie bracket $\{\cdot, \cdot\}$ satisfying $\{f, gh\} = g\{f, h\} + h\{f, g\}$ for all $f, g, h \in C^\infty(M)$. The latter condition, together with the antisymmetry of the bracket, is equivalent to the existence of a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ such that $\{f, g\} = \langle \Pi, df \otimes dg \rangle$.

Note that if one takes the antisymmetrization of this bracket (the original bracket Courant has introduced [10]), E does not become an algebroid in our sense, since eq. (1.4) will not be valid any more (as a consequence of the non-standard behavior of the above bracket under multiplication by a function of the section in the first entry).

Lemma 2.5 *The anchor map of a Loday algebroid is a morphism of brackets.*

Proof. Obviously, from (2.1) we find $[[\psi_1, \psi_2], f\psi_3] = [\psi_1, [\psi_2, f\psi_3]] - (\psi_1 \leftrightarrow \psi_2)$. Using the Leibniz rule for the l.h.s., we obtain $\rho([\psi_1, \psi_2])f\psi_3 + f[[\psi_1, \psi_2], \psi_3]$. Applying it twice to the first term on the r.h.s., we get $\rho(\psi_1)\rho(\psi_2)f\psi_3 + f[\psi_1, [\psi_2, \psi_3]] + \rho(\psi_1)f[\psi_2, \psi_3] + \rho(\psi_2)f[\psi_1, \psi_3]$. The last two terms drop out upon antisymmetrizing in ψ_1 and ψ_2 . In the remaining equation the terms proportional to f cancel by means of (2.1), and one is left with

$$\rho([\psi_1, \psi_2])f\psi_3 = [\rho(\psi_1), \rho(\psi_2)]f\psi_3, \quad (2.3)$$

valid for all sections ψ_i , $i = 1, 2, 3$, and functions f . This completes the proof. \square

Lemma 2.6 *The E-Lie derivative provides a representation of the bracket of a Loday algebroid on E-tensors, $[{}^E L_{\psi_1}, {}^E L_{\psi_2}] = {}^E L_{[\psi_1, \psi_2]}$.*

Proof. This follows from (2.1), the previous Lemma, and the extension of the E-Lie derivative to tensor powers and the dual by means of a Leibniz rule (using that commutators of operators satisfying a Leibniz rule for some algebra—here that of the tensor product as well as that of contractions—are of Leibniz type again). \square

A Lie algebroid permits to go further in extrapolating usual geometry on manifolds to the setting of more general vector bundles (algebroids). In particular, there is a straightforward generalization of the de Rham differential in precisely this case:⁷ In any almost Lie algebroid we may define the following degree one map ${}^E d$ on E -differential forms $\Omega_E(M) \equiv \Gamma(\Lambda^* E^*)$. For any function f and E -1-form ω we set

$$\langle {}^E d f, \psi \rangle := \rho(\psi)f, \quad \langle {}^E d \omega, \psi \otimes \psi' \rangle := \rho(\psi)\langle \omega, \psi' \rangle - \rho(\psi')\langle \omega, \psi \rangle - \langle \omega, [\psi, \psi'] \rangle \quad (2.4)$$

and extend this by means of a graded Leibniz rule to all of $\Omega_E(M)$. Clearly, for the standard Lie algebroid (TM, Id) this reduces to the ordinary de Rham differential. As one proves by induction, with this one finds in generalization of the Cartan-Koszul formula:

$$\begin{aligned} {}^E d \omega(\psi_1, \dots, \psi_{p+1}) &:= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\psi_i) \omega(\dots, \hat{\psi}_i, \dots) + \\ &+ \sum_{i < j} (-1)^{i+j} \omega([\psi_i, \psi_j], \dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots), \end{aligned} \quad (2.5)$$

⁷We remark in parenthesis that there is also an option to generalize the de Rham differential different from formula (2.4) below, using the language of graded manifolds; *that* generalization can be used also for Courant algebroids (cf. e.g. [32]) or even arbitrary L_∞ algebroids, cf. section 5 below.

valid for any $\omega \in \Omega_E^p(M)$ and $\psi_i \in \Gamma(M, E)$. This text is in part supposed to contain explicit proofs:

Proof. (of eq. (2.5)) The property holds for $p = 1$ by definition. Suppose we proved (2.5) for all forms of order at most $p - 1$. It suffices to see that (2.5) holds for each $\omega = \alpha \wedge \tau$, where $\alpha \in \Omega_E^1(M)$, $\tau \in \Omega_E^{p-1}(M)$. ${}^E\text{d}(\omega) = {}^E\text{d}(\alpha) \wedge \tau - \alpha \wedge {}^E\text{d}(\tau)$, therefore

$$\begin{aligned} {}^E\text{d}(\omega)(\psi_1, \dots, \psi_{p+1}) &= \sum_{i < j} (-1)^{i+j+1} {}^E\text{d}\alpha(\psi_i, \psi_j) \tau(\dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) + \\ &\sum_i (-1)^i \alpha(\psi_i) {}^E\text{d}\tau(\dots, \hat{\psi}_i, \dots) = \sum_{i < j} (-1)^{i+j+1} \left(\rho(\psi_i) \alpha(\psi_j) - \rho(\psi_j) \alpha(\psi_i) - \right. \\ &\alpha([\psi_i, \psi_j]) \left. \right) \tau(\dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) + \sum_i (-1)^i \alpha(\psi_i) \left(\sum_{j < i} (-1)^{j+1} \rho(\psi_j) \tau(\dots, \hat{\psi}_j, \dots, \hat{\psi}_i, \dots) \right. \\ &+ \sum_{i < j} (-1)^j \rho(\psi_j) \tau(\dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) + \sum_{\substack{k < l < i \\ i < k < l}} (-1)^{k+l} \tau([\psi_k, \psi_l], \dots, \widehat{\psi_{i,k,l}}, \dots) + \\ &\left. + \sum_{k < i < l} (-1)^{k+l+1} \tau([\psi_k, \psi_l], \dots, \widehat{\psi_{i,k,l}}, \dots) \right), \end{aligned}$$

Collecting all terms proportional to ρ , we find

$$\begin{aligned} &\sum_{i < j} (-1)^{i+j+1} \left(\rho(\psi_i) \alpha(\psi_j) - \rho(\psi_j) \alpha(\psi_i) \right) \tau(\dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) + \\ &\sum_i (-1)^i \alpha(\psi_i) \left(\sum_{j < i} (-1)^{j+1} \rho(\psi_j) \tau(\dots, \hat{\psi}_j, \dots, \hat{\psi}_i, \dots) + \sum_{i < j} (-1)^j \rho(\psi_j) \tau(\dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) \right) = \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\psi_i) \omega(\dots, \hat{\psi}_i, \dots), \end{aligned}$$

while the remaining terms yield

$$\begin{aligned} &\sum_{i < j} (-1)^{i+j} \alpha([\psi_i, \psi_j]) \tau(\dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) + \\ &\sum_i (-1)^i \alpha(\psi_i) \left(\sum_{\substack{k < l < i \\ i < k < l}} (-1)^{k+l} \tau([\psi_k, \psi_l], \dots, \widehat{\psi_{i,k,l}}, \dots) + \sum_{k < i < l} (-1)^{k+l+1} \tau([\psi_k, \psi_l], \dots, \widehat{\psi_{i,k,l}}, \dots) \right) = \\ &= \sum_{i < j} (-1)^{i+j} \omega([\psi_i, \psi_j], \dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots) \end{aligned}$$

which completes the proof. \square

One may now verify that the above operator squares to zero, ${}^E\text{d}^2 = 0$, iff (2.1) is satisfied (turning E into a Lie algebroid). In fact, *all* the information of a Lie algebroid is captured by ${}^E\text{d}$ as seen from the following

Proposition 2.7 *The structure of a Lie algebroid on a vector bundle $E \rightarrow M$ is in one-to-one correspondence with the structure of a differential complex on $(\Omega_E(M), \wedge)$.*

Proof. As obvious from (2.4), the skew-symmetric bracket on sections of E and the anchor map are uniquely determined by the canonical differential ${}^E d$. In particular, (2.4) can be rewritten as

$$\rho(\psi)f = [\iota_\psi, {}^E d]f, \quad (2.6)$$

$$\iota_{[\psi, \psi']} \omega = [\iota_\psi, [\iota_{\psi'}, {}^E d]] \omega, \quad (2.7)$$

where ι_ψ denotes the contraction with $\psi \in \Gamma(M, E)$ and $[\cdot, \cdot]$ the super bracket of super derivations of the graded commutative algebra $\Omega_E(M)$. Note that (2.7) has been verified for E -1-forms ω (and it is trivially satisfied if ω is a function); due to the required Leibniz property of ${}^E d$, however, it then necessarily holds for arbitrary $\omega \in \Omega_E(M)$ (which is generated by E -1-forms and functions). With these formulas we now find

$$\begin{aligned} (\rho([\psi_1, \psi_2]) - [\rho(\psi_1), \rho(\psi_2)])f &= \iota_{\psi_1} \iota_{\psi_2} {}^E d^2 f, \\ (\iota_{[\psi_1, [\psi_2, \psi_3]]} + c. p.)\omega &= \iota_{\psi_1} \iota_{\psi_2} \iota_{\psi_3} {}^E d^2 \omega - (\iota_{\psi_1} \iota_{\psi_2} {}^E d^2 (\iota_{\psi_3} \omega) + c. p.), \end{aligned}$$

valid for any $f \in C^\infty(M)$, $\omega \in \Omega_E^1(M)$, and $\psi_i \in \Gamma(M, E)$. Thus we see that the anchor map is a morphism of brackets if and only if the canonical differential squares to zero on functions, and the Jacobi condition for the almost Lie bracket holds if and only if ${}^E d^2 = 0$ on functions and E -1-forms simultaneously. This extends to all of $\Omega_E(M)$ since ${}^E d^2 = \frac{1}{2}[\iota_\psi, {}^E d]$ is a super derivative of the algebra $\Omega_E(M)$. Noting that the Leibniz rule follows from the derivative property of ${}^E d$ as well and that any Lie algebroid gives rise to such a differential finalizes the proof. \square

Eq. (2.7) shows that the bracket of a Lie algebroid is a so-called *derived bracket* [24], generalizing the wellknown formulas of Cartan: Eq. (2.6) generalizes the fact that on differential forms one has $L_\xi = [\iota_\xi, d]$ (while on a function the Lie derivative reduces to application of the vector field). Eq. (2.7) reduces to the evident identity $-\iota_{L_{\xi'}}(\xi) = [\iota_\xi, L_{\xi'}]$ for the standard Lie algebroid (using $L_{\xi'}(\xi) = -[\xi, \xi']$). We remark in parenthesis that in the case of E being a Lie algebroid the E -Lie derivative, defined for a general Loday algebroid, satisfies also the usual formula

$${}^E L_\psi = \iota_\psi {}^E d + {}^E d \iota_\psi \quad (2.8)$$

on E -differential forms. Later in this article we also show how the Courant (or Dorfmann) bracket can be put into the form of a derived bracket—for the case of eq. (2.2) already at the end of this section, while for the more general setting, which we are going to define in what follows, in section 5 below.

Definition 2.8 A Courant algebroid is a Loday algebroid $(E, \rho, [\cdot, \cdot])$ together with an invariant E -metric ${}^E g$ such that ${}^E g([\psi, \psi'], \psi') = \frac{1}{2}\rho(\psi'){}^E g(\psi, \psi)$.

We will mostly denote ${}^E g(\psi, \psi')$ simply as (ψ, ψ') , except if the appearance of the metric shall be stressed. The E -metric is used to control the symmetric part of the bracket. It permits to deduce the behavior of the bracket under multiplication of the first section w.r.t. a function for example:

$$[f\psi_1, \psi_2] = f[\psi_1, \psi_2] - (\rho(\psi_2)f)\psi_1 + (\psi_1, \psi_2)\rho^*(df). \quad (2.9)$$

In a general Loday algebroid there is *no* restriction to the bracket like this at all.

Invariance of ${}^E g$ means that for any section $\psi \in \Gamma(M, E)$ one has ${}^E L_\psi {}^E g = 0$; this is the same as requiring

$$([\psi_1, \psi_2], \psi_3) + (\psi_2, [\psi_1, \psi_3]) = \rho(\psi_1)(\psi_2, \psi_3) \quad (2.10)$$

for any $\psi_i \in \Gamma(M, E)$. When ρ , considered as a section of $\text{Hom}(E, TM)$, is nonzero at any point of M , remarkably the invariance can be also concluded from the remaining three axioms of a Courant algebroid cf. [18]. Also, it is easy to see that just Eq. (2.10) by itself permits to conclude the Leibniz identity (1.4): Replace ψ_2 by $f\psi_2$ to obtain $([\psi_1, f\psi_2], \psi_3) - f([\psi_1, \psi_2], \psi_3) = (\rho(\psi_1)f)(\psi_2, \psi_3)$, which yields the claimed equation by bilinearity and non-degeneracy of the inner product. So, any anchored vector bundle equipped with some bracket on its sections and an invariant fiber metric in the sense of (2.10) is an algebroid as defined in the introduction.

Sometimes in the literature an antisymmetrization of the above bracket $[\cdot, \cdot]$ is used in the definition, cf. e.g. the contribution of Hitchin to this volume;⁸ this antisymmetrized bracket, however, has *neither* of the two nice properties (1.4) and (2.1), for which reason we preferred the present version of axiomatization.

For M a point, the definition of a Courant algebroid is easily seen to reduce to a quadratic Lie algebra, i.e. to a Lie algebra endowed with a nondegenerate, ad-invariant inner product: Indeed, since then the r.h.s. of the last equation in definition 2.8 vanishes, the bracket becomes antisymmetric and (2.10) apparently reduces to the notion of ad-invariance of the metric. (Evidently in this context, or whenever ρ has zeros, the invariance needs to be required separately). The fact that Courant algebroids provide a generalization of quadratic Lie algebras may be one motivation for introducing them, since in particular there is also the following simple observation:

Proposition 2.9 *Let E be a Lie algebroid. Then it admits an invariant fiber metric ${}^E g$ only in the (very restrictive) case of $\rho \equiv 0$.*

Proof. The statement follows at once from the formula (deduced from the Leibniz property of the E -Lie derivative and formula (2.8) applied to E -1-forms)

$${}^E L_{f\psi} {}^E g = f {}^E L_\psi {}^E g + 2 {}^E df \vee \iota_\psi {}^E g, \quad (2.11)$$

valid for any $f \in C^\infty(M)$, $\psi \in \Gamma(M, E)$, where \vee denotes the symmetrized tensor product: according to the first formula (2.4), ${}^E df$ vanishes for all functions f only if $\rho \equiv 0$. \square

Nevertheless, the option (2.10) is not the only possibility to generalize quadratic Lie algebras to the realm of algebroids. There are at least two more, which will also play a role in an attempt to generalize Yang-Mills theories to the context of structural

⁸At least in the case of exact Courant algebroids with $H = 0$, introduced below, the antisymmetrized bracket is called the Courant bracket and the bracket used above the Dorfmann bracket. In the axiomatization of a general Courant algebroid, both types of brackets can be used, and, following [34], we prefer the non-antisymmetric bracket for the reason to follow. Although in the more recent literature the name Dorfmann bracket seems to prevail for the nonantisymmetric bracket in an exact Courant algebroid, we will often use the terminus ‘‘Courant bracket’’ for it.

algebroids/groupoids, as explained in the last section. We thus introduce these two, which are also interesting in their own right.

Using the notion of E -Lie derivatives and inspired by the philosophy to treat E as the tangent bundle of a manifold with the ensuing geometrical intuition, we consider

Definition 2.10 *Let E be a Loday algebroid. We call it a maximally symmetric E -Riemannian space if E is equipped with a definite fiber metric Eg which permits a (possibly overcomplete) basis of sections $\psi_\alpha \in \Gamma(E)$, $\langle \psi_\alpha(x) \rangle = E_x \ \forall x \in M$, $\alpha = 1 \dots s \geq \text{rank } E$, such that ${}^EL_{\psi_\alpha}{}^Eg = 0$.*

Likewise we can define a maximally symmetric E -pseudo-Riemannian space by dropping the condition of the definiteness of the fiber metric. For $E = TM$ this reduces to the standard notion of a maximally symmetric pseudo-Riemannian space (manifold), whereas for a Lie algebra (a Lie algebroid over a point) this reproduces the notion of quadratic Lie algebras.

So, in contrast to (2.10), *this* notion is compatible with a Lie algebroid with nonvanishing anchor, although it also poses restrictions (like it is already the case in the example of Lie algebras). Another option for generalizing quadratic Lie algebras to Lie algebroids is the following one using an algebroid type covariant derivative. For this we define

Definition 2.11 *Let $E \rightarrow M$ be an algebroid and $V \rightarrow M$ another vector bundle over the same base manifold. Then an E -covariant derivative ${}^E\nabla$ on V is a map from $\Gamma(E) \otimes \Gamma(V) \rightarrow \Gamma(V)$, $(\psi, v) \mapsto {}^E\nabla_\psi v$ that is $C^\infty(M)$ linear in the first entry and satisfies the Leibniz rule*

$${}^E\nabla_\psi(fv) = f {}^E\nabla_\psi v + \rho(\psi)f v \quad \forall f \in C^\infty(M). \quad (2.12)$$

Clearly, this notion reduces to an ordinary covariant derivative on V for the case of the standard Lie algebroid $E = TM$. Moreover, any ordinary connection ∇ on V gives rise to an E -covariant derivative by means of ${}^E\nabla_\psi := \nabla_{\rho(\psi)}$, while certainly not every E -connection is of this form (for $\rho \equiv 0$, e.g., this expression vanishes identically, while in this case *any* section of $E^* \otimes \text{End}(V)$ defines an E -connection). In the particular case of a Lie algebroid (and only there) one can generalize the notions of curvature and torsion to the algebroid setting, with ${}^ER \in \Omega_E^2 \otimes \Gamma(\text{End } V)$ and ${}^ET \in \Omega_E^2 \otimes \Gamma(E)$. Note that a “flat” E -connection, i.e. one with ${}^ER = 0$, is what one calls a *Lie algebroid representation* on V (generalizing the ordinary notion of a Lie algebra representation on a vector space, to which this reduces for the case of M being a point).

Given a fiber-metric on V it is called compatible with an E -connection on V , if it is annihilated by E -covariant derivatives. In generalization of the well-known fact of the uniqueness of a metric-compatible, torsion-free connection on a manifold ($E = TM$), also for a general Lie algebroid E one can prove that there is a unique E -torsion-free E -connection compatible with a (pseudo-)Riemannian metric Eg . On the other hand, given an ordinary connection ∇ on an algebroid E , we can also regard the following canonically induced E -connection

$$\widetilde{{}^E\nabla}_{\psi_1} \psi_2 = \nabla_{\rho(\psi_2)} \psi_1 + [\psi_1, \psi_2], \quad (2.13)$$

which, for E a Lie algebroid, differs from ${}^E\nabla_\psi := \nabla_{\rho(\psi)}$ by subtraction of its own E -torsion. It is easy to see that a Lie algebroid E with fiber metric Eg compatible with $\widetilde{{}^E\nabla}$, induced by some ordinary connection on E , reduces to a quadratic Lie algebra; here one may or may not want to impose that this E -connection is flat, while only in the first case one would call it an E -metric invariant under the adjoint *representation* induced by the connection ∇ .

Before discussing the second motivation of introducing Courant algebroids, which will lead us into the world of Dirac and generalized complex structures, we use the opportunity of having introduced E -covariant derivatives for a natural generalization of Courant algebroids:⁹

Definition 2.12 *A vector bundle twisted or V -twisted Courant algebroid is given by a Loday algebroid structure $(E, \rho, [\cdot, \cdot])$ on $E \rightarrow M$ together with a second vector bundle $V \rightarrow M$, an E -covariant derivative ${}^E\nabla$ on V , and a surjective, non-degenerate bilinear map $(\cdot, \cdot): E \times_M E \rightarrow V$ such that*

$$([\psi, \psi], \psi') = \frac{1}{2} {}^E\nabla_{\psi'}(\psi, \psi) = ([\psi', \psi], \psi). \quad (2.14)$$

It is not difficult to see that for the case of V being a trivial \mathbb{R} -bundle over M , admitting an E -covariantly constant basis section, the above definition reduces to the one of an ordinary Courant algebroid. A V -twisted Courant algebroid defined over a point does no more reduce to Lie algebras, but rather gives a Leibniz algebra E only, with V being an E -module (cf. [17] for further details on these statements). V -twisted Courant algebroids play a role in the context of higher gauge theories, in particular nonabelian gerbes.

For later use we also recall the definition of a (strict) Lie 2-algebroid from [17]:

Definition 2.13 *A strict Lie 2-algebroid are two Lie algebroids over M , $(E \rightarrow M, \rho, [\cdot, \cdot])$ as well as $(V \rightarrow M, 0, [\cdot, \cdot]_V)$ together with a morphism $t: V \rightarrow E$ and a representation ${}^E\nabla$ of E on V such that*

$${}^E\nabla_{t(v)}w = [v, w]_V, \quad t({}^E\nabla_\psi v) = [\psi, t(v)], \quad \forall v, w \in \Gamma(V), \forall \psi \in \Gamma(E). \quad (2.15)$$

Note that the second Lie algebroid V is a bundle of Lie algebras only, $\rho_V \equiv 0$. The general notion of a morphism for Lie algebroids is somewhat involved but most easily defined in terms of the Q-language developed later (and thus provided in Sec. 5 below); over the same base manifold as here, however, it simply implies $t([v, w]) = [t(v), t(w)]$ as well as $\rho \circ t = \rho_V$. For M being a point, the above definition reduces to the one of a strict Lie 2-algebra or, equivalently, a differential crossed module. The above definition may be twisted by an E -3-form H taking values in V , cf. Theorem 3.1 in [17], in which case one obtains what one might call an H -twisted or semistrict Lie 2-algebroid or simply a Lie 2-algebroid. (As one of the names suggests, it reduces to a semistrict Lie 2-algebra when M is a point. The last name is most natural in view of the relation to so-called Q-manifolds, cf. section 5 below).

⁹This notion was introduced independently and with different, but complementary motivations in [17] and [9].

We now return to ordinary Courant algebroids. The other motivation for considering them comes from the quest for generalizing the notions of (pre-) symplectic, Poisson, and complex manifolds to so-called (real and/or complex) Dirac structures, our subject in the next section. For their definition one restricts to particular kinds of Courant algebroids, so called exact Courant algebroids: For a general Courant algebroid one obviously has the following sequence of vector bundles

$$0 \rightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \rightarrow 0, \quad (2.16)$$

where ρ^* is the fiberwise transpose of ρ combined with the isomorphism induced by E_g .¹⁰

Definition 2.14 *An exact Courant algebroid is a Courant algebroid such that the sequence (2.16) is exact.*

For the rest of this section E will always denote an exact Courant algebroid.

Proposition 2.15 *The image of T^*M in E is a maximally isotropic subbundle w.r.t. E_g .*

Proof. By the definition of ρ^* , $(\rho^*\omega, \psi) = \langle \omega, \rho(\psi) \rangle$ therefore $(\omega_1, \omega_2) = 0$ for all $\omega_i \in \Omega^1(M)$. Thus $\rho^*T^*M \subset (\rho^*T^*M)^\perp$; by a dimensional argument, using that the E -metric is nondegenerate, we may conclude equality.

Definition 2.16 *A splitting (also sometimes called a connection) of an exact Courant algebroid is a map $j: TM \rightarrow E$, such that $\rho \circ j = \text{id}$ and $j(TM)$ is isotropic.*

Evidently $j(TM)$ defines a maximally isotropic subbundle of E complementary to T^*M . Hereafter (within this and the following two sections) we identify a 1-form with its image in the exact Courant algebroid and, given a splitting, we can likewise do so for the image of vector fields w.r.t. the splitting. A general element of E can thus be written as the sum of an element of T^*M and TM , provided a splitting is given. Existence of the latter one is guaranteed by the following

Lemma 2.17 *There always exists a splitting in E . The set of splittings is a torsor over $\Omega^2(M)$.*

Proof. Let us take any splitting of the exact sequence $0 \rightarrow T^*M \rightarrow E \rightarrow TM \rightarrow 0$ in the category of vector bundles, $j: TM \rightarrow E$. Since T^*M is maximally isotropic, the pairing between $j(TM)$ and T^*M has to be nondegenerate. Therefore for any vector field ξ there exists a unique 1-form $\beta(\xi)$ such that $(j(\xi), j(\xi')) = \langle \beta(\xi), \xi' \rangle$. The new splitting given by $j(\xi) - \frac{1}{2}\beta(\xi)$ is maximally isotropic (note that for any 1-form $\alpha = \beta(\xi)$ one has by definition of ρ^* and j : $(\rho^*(\alpha), j(\xi')) = \langle \alpha, (\rho \circ j)(\xi') \rangle = \langle \alpha, \xi' \rangle$). Any other splitting differs from the chosen one by a section B of $T^*M \otimes T^*M$, i.e. by sending

$$j_B(\xi) = j(\xi) + B(\xi, \cdot). \quad (2.17)$$

The tensor field B is necessarily skew-symmetric because the image of TM is required to be maximally isotropic. \square

¹⁰In fact, this even defines a complex of sheaves, since for a general Courant algebroid $\rho \circ \rho^* \equiv 0$, cf. e.g. [10].

Proposition 2.18 For each $\omega \in \Omega^1(M)$ and $\psi \in \Gamma(E)$ one has:

$$[\psi, \rho^* \omega] = \rho^* L_{\rho(\psi)} \omega, \quad [\rho^* \omega, \psi] = -\rho^* \iota_{\rho(\psi)} d\omega. \quad (2.18)$$

Proof. Invariance of the E -metric implies

$$\rho(\psi_1)(\rho^* \omega, \psi_2) = ([\psi_1, \rho^* \omega], \psi_2) + (\rho^* \omega, [\psi_1, \psi_2]),$$

hence $([\psi_1, \rho^* \omega], \psi_2) = \langle L_{\rho(\psi_1)} \omega, \rho(\psi_2) \rangle$, where we used $\rho([\psi_1, \psi_2]) = [\rho(\psi_1), \rho(\psi_2)]$. The E -metric being nondegenerate, we may conclude the first rule of multiplication. The second one follows from this by the symmetry property of the bracket in a Courant algebroid: $[\psi, \psi] = \rho^* d(\psi, \psi)$. \square

A direct consequence is:

Corollary 2.19 The image of T^*M is an abelian ideal of E .

A result in the classification of exact Courant algebroids, ascribed to P. Severa [33], is the following one:

Proposition 2.20 Up to isomorphism there is a one-to-one correspondence between exact Courant algebroids and elements of $H^3(M, \mathbb{R})$. In particular, with the choice of a splitting of (2.16) it takes the form

$$[\xi + \alpha, \xi' + \alpha'] = [\xi, \xi'] + H(\xi, \xi', \cdot) + L_\xi \alpha' - \iota_{\xi'} d\alpha, \quad (2.19)$$

where $H \in \Omega^3(M)$ and $dH = 0$. For a change of splitting parameterized by a two-form B as in (2.17), the 3-form transforms as $H \mapsto H + dB$.

Proof. Given a splitting of (2.16), one has $[j(\xi), j(\xi')] = j([\xi, \xi']) + H(\xi, \xi')$, with H the “curvature” of the splitting, taking values in 1-forms. The Leibnitz property of the Courant bracket applied to the formula above implies that H is $C^\infty(M)$ -linear with respect to its first two arguments, hence H can be identified with a section of $(T^*M)^{\otimes 3}$. The image of TM by j is an isotropic subbundle of E . Therefore the Courant bracket restricted to $j(TM)$ is skew-symmetric and therefore also the curvature H , i.e. $H(\cdot, \cdot)$. Moreover, using that the Courant metric is invariant w.r.t. the Courant bracket and that the image of j is isotropic, we obtain

$$\langle H(\xi_1, \xi_2), \xi_3 \rangle = ([j(\xi_1), j(\xi_2)], j(\xi_3)) = -(j(\xi_2), [j(\xi_1), j(\xi_3)]) = -\langle H(\xi_3, \xi_2), \xi_1 \rangle,$$

thus H is totally skew-symmetric and can be regarded as a 3-form. By the formula (2.18) the bracket of two sections of E with the splitting j is indeed found to take the form of (2.19).

Let us calculate the Jacobiator of sections $J(\psi_1, \psi_2, \psi_3) = [\psi_1, [\psi_2, \psi_3]] - [[\psi_1, \psi_2], \psi_3] - [\psi_2, [\psi_1, \psi_3]]$ for $\psi_i \in \Gamma(E)$, which has to vanish by the property of a Courant algebroid. Again we want to be explicit here. Using $\psi_i = \xi_i + \alpha_i$, where ξ_i are vector fields and α_i 1-forms, we obtain

$$\begin{aligned} [\psi_1, [\psi_2, \psi_3]] &= [\xi_1 + \alpha_1, [\xi_2, \xi_3] + H(\xi_2, \xi_3, \cdot) + L_{\xi_2} \alpha_3 - \iota_{\xi_3} d\alpha_2] = [\xi_1, [\xi_2, \xi_3]] + \\ &\quad + H(\xi_1, [\xi_2, \xi_3], \cdot) + L_{\xi_1} H(\xi_2, \xi_3, \cdot) + L_{\xi_1} L_{\xi_2} \alpha_3 - L_{\xi_1} \iota_{\xi_3} d\alpha_2 - \iota_{[\xi_2, \xi_3]} d\alpha_1, \end{aligned}$$

$$[[\psi_1, \psi_2], \psi_3] = \left[[\xi_1, \xi_2] + H(\xi_1, \xi_2, \cdot) + L_{\xi_1} \alpha_2 - \iota_{\xi_2} d\alpha_1, \xi_3 + \alpha_3 \right] = [[\xi_1, \xi_2], \xi_3] + \\ + H([\xi_1, \xi_2], \xi_3, \cdot) + L_{[\xi_1, \xi_2]} \alpha_3 - \iota_{\xi_3} dH(\xi_1, \xi_2, \cdot) - \iota_{\xi_3} L_{\xi_1} d\alpha_2 + \iota_{\xi_3} L_{\xi_2} d\alpha_1 ,$$

$$[\psi_2, [\psi_1, \psi_3]] = \left[\xi_2 + \alpha_2, [\xi_1, \xi_3] + H(\xi_1, \xi_3, \cdot) + L_{\xi_1} \alpha_3 - \iota_{\xi_3} d\alpha_1 \right] = [\xi_2, [\xi_1, \xi_3]] + \\ + H(\xi_2, [\xi_1, \xi_3], \cdot) + L_{\xi_2} H(\xi_1, \xi_3, \cdot) + L_{\xi_2} L_{\xi_1} \alpha_3 - L_{\xi_2} \iota_{\xi_3} d\alpha_1 - \iota_{[\xi_1, \xi_3]} d\alpha_2 ,$$

so one can rewrite the Jacobiator as

$$\begin{aligned} J(\psi_1, \psi_2, \psi_3) &= [\xi_1, [\xi_2, \xi_3]] - [[\xi_1, \xi_2], \xi_3] - [\xi_2, [\xi_1, \xi_3]] \\ &+ L_{\xi_1} L_{\xi_2} \alpha_3 - L_{[\xi_1, \xi_2]} \alpha_3 - L_{\xi_2} L_{\xi_1} \alpha_3 \\ &+ -L_{\xi_1} \iota_{\xi_3} d\alpha_2 + \iota_{\xi_3} L_{\xi_1} d\alpha_2 + \iota_{[\xi_1, \xi_3]} d\alpha_2 \\ &+ -\iota_{[\xi_2, \xi_3]} d\alpha_1 - \iota_{\xi_3} L_{\xi_2} d\alpha_1 + L_{\xi_2} \iota_{\xi_3} d\alpha_1 \\ &+ (dH)(\xi_1, \xi_2, \xi_3, \cdot). \end{aligned}$$

Apparently, the first four lines vanish and therefore the 3-form H entering into the product formula has to be closed. Finally, by a simple calculation one obtains

$$[\psi_1 + B(\rho(\psi_1), \cdot), \psi_2 + B(\rho(\psi_2), \cdot)] = [\psi_1, \psi_2] + dB(\rho(\psi_1), \rho(\psi_2), \cdot) ,$$

which finalizes the proof of the proposition 2.20. \square

Suppose a splitting of (2.16) is chosen. Then, each section of E , $\psi = \xi + \alpha$, can be thought of as an operator acting on differential forms by the formula

$$c(\psi) \omega = \iota_{\xi} \omega + \alpha \wedge \omega . \quad (2.20)$$

It is useful to consider the space of forms as a spinor module over the Clifford algebra of E with the quadratic form given by the Courant metric. The next simple proposition, the proof of which we leave for readers, shows that the notion of derived brackets can be exploited also in the case of Courant algebroids.

Proposition 2.21 *The following identity holds true:*

$$c([\psi_1, \psi_2]) = [[c(\psi_1), d + H], c(\psi_2)] . \quad (2.21)$$

3 Dirac structures

Definition 3.1 *Suppose E is an exact Courant algebroid over M and D a maximal totally isotropic (maximally isotropic) subbundle of E with respect to the Courant scalar product. Then D is called a Dirac structure, if the Courant bracket of two sections of D is again a section of D .*

Example 3.2 *T^*M is a Dirac structure, see Proposition 2.15 and Corollary 2.19. Given a connection in E (see Definition 2.16) TM is a maximally isotropic subbundle of E ; TM is a Dirac structure, iff the curvature H is zero.*

Suppose a splitting of a Courant algebroid E is chosen, $E = T^*M \oplus TM$, the curvature H of which is zero. We then have the following two examples showing that Poisson and presymplectic geometry give rise to particular Dirac structures.

Example 3.3 *Let D be a graph of a tensor field $\Pi \in \Gamma(\otimes^2 TM)$, considered as a map from T^*M to TM . Then D is a Lagrangian subbundle iff Π is skew-symmetric, i.e. iff it is a bivector field, $\Pi \in \Gamma(\Lambda^2 TM)$. The projection of D to T^*M is non-degenerate. Vice versa, any Lagrangian subbundle of E which has a non-degenerate projection to T^*M is a graph of a bivector field. The graph of a bivector field D is a Dirac structure if and only if Π is a Poisson bivector, $[\Pi, \Pi] = 0$.*

Example 3.4 *Let D be a graph of a 2-form $\omega \in \Omega^2(M)$ considered as a skew-symmetric map from TM to T^*M , then D is a Lagrangian subbundle and the projection of D to TM is non-degenerate. Any Lagrangian subbundle of E which has a non-degenerate projection to TM is a graph of a 2-form. D is a Dirac structure if and only if ω is closed.*

In the second example we could also have started with a general $(2,0)$ -tensor-field certainly, completely parallel to the first example. For the case that H is non-zero, one obtains twisted versions of both examples, cf. Eq. (1.3) in the case of example 3.3.

The complexification of E , $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$, is a real bundle, i.e. a bundle over \mathbb{C} endowed with a \mathbb{C} -anti-linear operator σ acting on E^c , such that $\sigma^2 = 1$ and $E = \ker(\sigma - 1)$. We next turn to a description of the algebraic set of complex subbundles of E^c , which are maximally isotropic with respect to the complexified Courant scalar product. In order to do so, we introduce an additional structure, namely a particular kind of fiber metric ${}^E g' =: g_\tau$ (cf. also [25]), different from the canonical one ${}^E g$, called sometimes a generalized Riemannian metric. Note that according to the philosophy presented here, a generalized Riemannian metric on a Courant algebroid would be any positive definite fiber metric; we will still adopt this partially established terminology now and then show that, given a splitting, it corresponds to an ordinary Riemannian metric g on M together with a 2-form B (cf. [25] as well as the contribution of N. Hitchin within this volume).

Definition 3.5 *A positive E -metric g_τ which can be expressed via an operator $\tau \in \Gamma(\text{End}(E))$ squaring to the identity such that for any $\psi_i \in \Gamma(E)$*

$$g_\tau(\psi_1, \psi_2) = (\tau\psi_1, \psi_2)$$

is called a generalized Riemannian metric.

It follows from the definition, that τ has to be self-adjoint with respect to the Courant metric ${}^E g(\cdot, \cdot) \equiv (\cdot, \cdot)$; because of $\tau^2 = 1$, it is orthogonal moreover. From $g_\tau(\psi, \psi) \geq 0$ we conclude that the $+1$ and -1 eigen-subspaces of τ are positive and negative definite, respectively, again with respect to the canonical metric in E , while $(\tau\psi_1, \tau\psi_2) = (\psi_1, \psi_2)$ shows that they are orthogonal to one another. As a corollary, ${}^E g$ having signature (n, n) , the dimensions of the two eigenvalue subspaces are equal. Vice versa, let us take any positive definite subbundle V of maximal rank, then there exists a unique τ which is

postulated to be 1 on V and -1 on the orthogonal subbundle V^\perp (with respect to the canonical metric). The operator τ satisfies the properties of the definition as above.

Given a splitting of the exact sequence (2.16), there is a one-to-one correspondence between generalized Riemannian metrics in the sense of the Definition 3.5 and the one of Hitchin within this volume. Since V is positive, it has zero intersection with any Lagrangian subspace of E and in particular with TM and T^*M (i.e. with $j(TM) \cong TM$ and $\rho^*(T^*M) \cong T^*M$). Hence V is a graph of an invertible bundle map $TM \rightarrow T^*M$ which can be identified with a non-degenerate tensor in $\Gamma(T^*M \otimes T^*M)$ such that its symmetric component g in the decomposition into the symmetric and antisymmetric parts, $g + B$, is a Riemannian metric on TM .

Define a new real form by using the complexification of τ : $\tau_\sigma := \sigma\tau$. Since τ is a real operator, its complexification commutes with the complex conjugation σ , therefore $\tau_\sigma^2 = 1$. Moreover, τ_σ is an anti-linear operator as a composition of linear and anti-linear operators, thus it defines a new real structure in E^c (different from the old one σ). Hereafter we shall distinguish the σ - and τ_σ -real forms by calling them "real" and " τ -real", respectively.

Proposition 3.6 *The real and imaginary subbundles of E^c with respect to the real structure τ_σ are the subbundles (over \mathbb{R}) $E^+ := V \oplus iV^\perp$ and $E^- := V^\perp \oplus iV$, respectively. Any complex Dirac structure in E^c is a totally complex subbundle w.r.t. τ_σ , i.e. its intersections with the τ_σ -real and totally complex subbundles of E^c are trivial.*

Proof. The first statement follows trivially from the definition of τ_σ : one needs to take into account that τ_σ is anti-linear. To check the second statement, it suffices to notice that the (real) eigen-subspaces of the new real structure τ_σ , E^+ and E^- , are positive (negative) definite with respect to the restriction of the (complexified) Courant metric in E^c , therefore E^\pm do not contain any nonzero isotropic vectors. \square

Let D be a maximally isotropic complex subbundle of E^c ; by the proposition above, D is a totally complex subbundle as well as $\tau_\sigma(D)$. Define a linear complex structure J in E by requiring $J(D) = i$ and $J(\tau_\sigma(D)) = -i$. By construction, J is a τ -real operator. Moreover, the restriction of J to E^+ is an orthogonal operator with respect to the induced positive metric in E^+ .

Proposition 3.7 *There is a one-to-one correspondence between complex maximally isotropic subbundles in E^c and \mathbb{R} -linear orthogonal operators J in E^+ which satisfy $J^2 = -1$. In particular, if D is a real maximally isotropic subbundle, then J is uniquely represented by a real orthogonal operator $S : V \rightarrow iV^\perp$ as follows:*

$$J_\sigma : \begin{pmatrix} 0 & -S^{-1} \\ S & 0 \end{pmatrix}. \quad (3.1)$$

Proof. We need to check only the last statement. Suppose D is a complexification of some real maximally isotropic subbundle of E , then D is preserved by the real structure σ and $\tau_\sigma(D) = \tau(D)$. Now, by construction, J anti-commutes with σ . Let us identify E^+

with $V \oplus V^\perp$ such that σ acts as $+1$ on the first and as -1 on the second factors. This identification is an isometry, if we supply V^\perp with an opposite metric (the original one on V^\perp is negatively definite). Since J anti-commutes with σ , it has only off-diagonal entries. Now, taking into account that J is orthogonal and squares to -1 , we immediately get the required form (3.1). \square

Proposition 3.8 *Let D be a maximally isotropic real subbundle of E provided with the metric induced by g_τ . Then the projector of D to V and V^\perp is an isometry up to a factor $\frac{1}{2}$ and $-\frac{1}{2}$, respectively.*

Proof. Taking into account that the orthogonal projector to V and V^\perp can be expressed as $P_V = \frac{1}{2}(1 + \tau)$ and $P_V^\perp = \frac{1}{2}(1 - \tau)$, respectively, we obtain:

$$\begin{aligned} (P_V \psi_1, P_V \psi_2) &= \frac{1}{2} ((\tau \psi_1, \psi_2) + (\psi_1, \psi_2)) \\ (P_V^\perp \psi_1, P_V^\perp \psi_2) &= \frac{1}{2} (-(\tau \psi_1, \psi_2) + (\psi_1, \psi_2)) \end{aligned}$$

for all $\psi_1, \psi_2 \in \Gamma(E)$. Imposing $\psi_i \in \Gamma(D)$, we get the required property. \square

Let us take $D = T^*M$ supplied with the positive metric, induced by g_τ in E as above.

Corollary 3.9 *There is a one-to-one correspondence between real maximally isotropic subbundles and orthogonal operators acting point-wisely in TM .*

Proof. The composition of maps $P_V^\perp \circ (P_{V|D})^{-1} : V \rightarrow V^\perp$ is an isometry up to a factor -1 (an anti-isometry). Combining this map with the natural anti-isometry of real spaces, $V^\perp \rightarrow iV^\perp$, given by the multiplication with i , we get an isometric identification of V and iV^\perp , which allows to identify S with an orthogonal operator acting in V . Using the conjugation by P_V , we identify S with an orthogonal operator acting in T^*M or, by the duality, in TM . \square

The above operator S is a section of $O(TM)$, the associated bundle $P \times_{O(n)} O(n)$ where P is the bundle of orthogonal frames and $O(n)$ acts on itself by conjugation. As it was constructed, the homotopy class of the section S depends only on the homotopy class of the corresponding maximally isotropic subbundle. One knows that, for a Lie group G , any principal G -bundle over M is the pull-back by the canonical one over the universal classifying space, $EG \rightarrow BG$ with respect to some map $\varphi_0 : M \rightarrow BG$. Then $P \times_G Y$ is the pull-back of $EG \times_G Y$ for each G -space Y . Applying this to $G = Y = O(n)$, one has the following commutative diagram:

$$\begin{array}{ccc} O(TM) & \xrightarrow{\varphi} & EO(n) \times_{O(n)} O(n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi_0} & BO(n) \end{array} \quad (3.2)$$

Given any section $S : M \rightarrow O(TM)$, the pull-back of $\varphi \circ S$ defines a map¹¹ from the equivariant cohomology of $O(n)$ with coefficients in some ring \mathcal{R} to the cohomology of

¹¹This construction was suggested by A. Alekseev during our joint work on Dirac structures.

M :

$$(\varphi \circ S)^* : H^i(EO(n) \times_{O(n)} O(n), \mathcal{R}) = H^i(O(n), \mathcal{R})^{O(n)} \rightarrow H^i(M, \mathcal{R}) . \quad (3.3)$$

Since all the arrows are defined up to homotopy, we obtain a characteristic map from the product of the set of homotopy classes of maximally isotropic subbundles with $H^i(O(n), \mathcal{R})$ to the cohomology of M . The explicit construction of this characteristic map for $\mathcal{R} = \mathbb{R}$ can be done by use of the secondary characteristic calculus. Let ∇ be the Levi-Civita connection of the induced metric, Φ an ad-invariant polynomial on the Lie algebra $\mathfrak{o}(n)$, then define the corresponding characteristic class as follows:

$$c_\Phi(S, g) = \int_0^1 \Phi(dt \wedge A + F(t)) , \quad (3.4)$$

where $A = S^{-1}\nabla(S)$, and $F(t)$ is the curvature of $\nabla + tA$. One can easily check that

$$dc_\Phi(S, g) = \Phi\left((S^{-1} \circ \nabla \circ S)^2\right) - \Phi(R) = 0 ,$$

where R is the curvature of ∇ (the Riemann curvature), since Φ is ad-invariant. The cohomology class of $c_\Phi(S, g)$ does not change under the homotopy of S and g .

Proposition 3.10 *Let D be a maximally isotropic subbundle of E . Suppose that there is a connection, i.e. an isotropic splitting of $0 \rightarrow T^*M \rightarrow E \rightarrow TM \rightarrow 0$, such that the projection of D to TM or T^*M is non-degenerate. Then the section S , constructed as above, is homotopic to 1 and -1 , respectively, thus all characteristic classes vanish.*

Proof. Obviously, whatever metric g is taken, T^*M corresponds to $-1 \in \Gamma(O(TM))$. Assume that the projection of D to T^*M is non-degenerate, then, apparently, D is homotopic to T^*M and thus the section S , corresponding to D , is homotopic to 1. If the projection of D to TM is non-degenerate, then D is homotopic to image TM and thus to any maximally isotropic subbundle with zero intersection with T^*M . Now it suffices to take the orthogonal complement of T^*M in E with respect to the generalized Riemann metric g_τ : this is a maximally isotropic subbundle of the required type which corresponds to 1. \square

Proposition 3.11 *Let D and D' be maximally isotropic subbundles corresponding to S and $S' \in \Gamma(O(T^*M))$, respectively, then the intersection $D \cap D'$ is zero if and only if $S^{-1}S' - 1$ is non-degenerate.*

Proof. The proof follows from the explicit parameterization of D by T^*M :

$$D_x = \{(1 + \tau)\eta + (1 - \tau)S\eta \mid \eta \in T_x^*M\} . \quad (3.5)$$

\square

Corollary 3.12 *D and T^*M admit a common complementary maximal isotropic subbundle, if and only if there exists $R \in \Gamma(O(TM))$ such that $R - 1$ and $RS - 1$ are non-degenerate.*

Previously we proved that if a Lagrangian subbundle D and T^*M has a common complementary maximally isotropic subbundle, then the operator S which corresponds to D can be deformed to the minus identity section (we take the connection in E such that the image of TM coincides with the chosen common complementary subbundle). In general, we can not reverse this statement, i.e. even if D is homotopic to T^*M^\perp , there may be no splitting in E such that the projection of D to T^*M becomes non-degenerate.

Example 3.13 *Let M be a 2-dimensional surface, then $\det S = \det R = 1$. Therefore both operators S and R can be thought of as points in S^1 .*

Let us assume that M contains a loop such that S passes $1 \in S^1$ and covers S^1 many times in both directions. We only claim that the total degree of the map from the loop to S^1 equals to zero (hence S is homotopically equivalent to the identity map). For example, $M = T^2$. One can take a smooth function $\phi(t) = a \sin(2\pi it)$. The quotient $t \mapsto \phi(t) \bmod \mathbb{Z}$ defines a smooth map $S^1 \rightarrow S^1$ of degree zero, because the limit of $\phi(t)$ at $a \rightarrow 0$ is zero. Nevertheless, the image of ϕ wraps S^1 many times for sufficiently large a . Now we can trivially extend the map to T^2 which gives S . Since the function, which would correspond to R , is not permitted to coincide with ϕ and to pass through 1, we conclude that there is “no room” for R .

4 Generalized complex structures

Generalized complex structures (GCS) were invented by Nigel Hitchin [20]. It turned out that GCS provides a mathematical background of certain sigma models (for instance, those the target space of which is endowed with a bihermitian metric, cf. [14, 27]). Generalized complex structures interpolate naturally between symplectic and holomorphic Poisson geometry.

Definition 4.1 *Let E be an exact Courant algebroid. A generalized complex structure is a maximally isotropic pure totally complex Dirac subbundle of the complexification of E with the complexified Courant scalar product, that is, a maximally isotropic subbundle D of $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$ such that D is closed w.r.t. the Courant bracket and $D \cap \overline{D} = \{0\}$.¹²*

As in Section 3, we uniquely associate a maximally isotropic totally complex subbundle D of E^c with a real point-wisely acting operator J which squares to -1_E , such that, at any $x \in M$ the fibers D_x and \overline{D}_x are the $+i$ and $-i$ eigen-subspaces of J_x , respectively. The isotropy condition of D implies that J is skew-symmetric with respect to the Courant scalar product (\cdot, \cdot) . From now on we shall consider only an “untwisted” version of an exact Courant algebroid together with an isomorphism $E \simeq TM \oplus T^*M$, where the latter direct sum is supplied with the canonical scalar product and the (Dorfman) bracket (2.2). We also identify bivector fields and 2-forms on M with sections of $\text{Hom}(T^*M, TM)$ and $\text{Hom}(TM, T^*M)$, respectively, by use of the corresponding contractions. The next lemma gives a complete set of algebraic conditions for J .

¹²We call such a subbundle D totally complex because its “real part” $D \cap \overline{D}$ is zero

Lemma 4.2 *J is an operator of the form*

$$J = \begin{pmatrix} I & \Pi \\ \Omega & -I^* \end{pmatrix}, \quad (4.1)$$

where $I \in \Gamma(\text{End}TM)$, $\Pi \in \Gamma(\Lambda^2 TM)$, and $\Omega \in \Omega^2(M)$. Then $J^2 = -1_E$ translates into:

$$\begin{aligned} I^2 + \Pi\Omega &= -1_{TM} \\ (I^*)^2 + \Omega\Pi &= -1_{T^*M} \\ I^*\Omega - \Omega I &= 0 \\ I\Pi - \Pi I^* &= 0 \end{aligned} \quad (4.2)$$

Proof. Straightforward calculation. \square

Similarly to the end of Section 2, we treat differential forms on M as sections of the spin module over the Clifford bundle $C(E)$, that is, the bundle of associative algebras generated by E subject to the relations $\varphi^2 = (\varphi, \varphi)$ for all $x \in M$ and $\varphi \in E_x$. The spinor action of $\Gamma(C(E))$ on $\Omega^*(M)$ is the extension of (2.20). Taking into account that the generalized complex structure operator J is skew-symmetric with respect to the scalar product, it can be thought of as a section of $C(E)$, which we denote by the same letter J , such that $J(\psi) = [J, \psi]$ for each $\psi \in \Gamma(E)$. Here $[\cdot, \cdot]$ is the (super-)commutator of sections of the Clifford bundle. It is easy to check that

$$\mathbf{c}(J)(\alpha_1 \wedge \dots \wedge \alpha_p) = - \sum_{i=1}^p \alpha_1 \wedge \dots \wedge I^*(\alpha_i) \wedge \dots \wedge \alpha_p + (\iota_\Pi + \Omega \wedge) \alpha_1 \wedge \dots \wedge \alpha_p, \quad (4.3)$$

where $\alpha_i \in \Omega^1(M)$, $i = 1, \dots, p$.

We now focus on the question when a maximally isotropic totally complex subbundle D is closed w.r.t. the Courant bracket or, in other words, when it is integrable.

Proposition 4.3 *D is a generalized complex structure if and only if for any $\psi, \psi' \in \Gamma(E)$ one has*

$$J([J\psi, \psi'] + [\psi, J\psi']) + [\psi, \psi'] - [J\psi, J\psi'] = 0. \quad (4.4)$$

Proof. The l.h.s. of the equation (4.4) is bilinear and real, thus it is sufficient to check the property for $\psi_1, \bar{\psi}_2$ and ψ_1, ψ_2 , where $\psi_i \in \Gamma(D)$. While in the first case the expression vanishes identically, the second one gives

$$J([J\psi_1, \psi_2] + [\psi_1, J\psi_2]) + [\psi_1, \psi_2] - [J\psi_1, J\psi_2] = 2i(J[\psi_1, \psi_2] - i[\psi_2, \psi_2]),$$

which is identically zero if and only if $[\psi_1, \psi_2]$ is again a section of D . The latter is nothing else but the integrability condition for D .

\square

The meaning of equation (4.4) is the same as for an almost complex structure in the usual sense, that is, the vanishing of a certain tensor called the (generalized) Nijenhuis tensor of J . The next Lemma provides an explicit construction of the Nijenhuis tensor in terms of the spin module.

Lemma 4.4 *The operator $N_J: = \frac{1}{2}([\mathbf{c}(J), [\mathbf{c}(J), \mathbf{d}]] + \mathbf{d})$ is a point-wisely acting map, such that for any $\psi, \psi' \in \Gamma(E)$ the following identity holds:*

$$[\mathbf{c}(\psi'), [\mathbf{c}(\psi), N_J]] = \mathbf{c}(J([J\psi, \psi'] + [\psi, J\psi']) + [\psi, \psi'] - [J\psi, J\psi']) .$$

Proof. The first part of the Lemma follows from the identity

$$2[N_J, f] = [[\mathbf{c}(J), [\mathbf{c}(J), \mathbf{d}]], f] + [\mathbf{d}, f] = \mathbf{c}(J^2(df) + \mathbf{d}f) = 0 ,$$

which holds for each $f \in C^\infty(M)$. The second part requires a simple computation which essentially based upon the derived formula of the Courant bracket (2.21) when $H = 0$. Indeed, let us write the Nijenhuis tensor in the form $N_J = \frac{1}{2}(ad_J^2(\mathbf{d}) + \mathbf{d})$ where $ad_J(a): = [\gamma_J, a]$ for any a . It is clear that

$$[\mathbf{c}(\psi), ad_J^2(a)] = -[\mathbf{c}(\psi), a] - 2ad_J([\mathbf{c}(J\psi), a]) + ad_J^2([\mathbf{c}(\psi), a]) .$$

Thus, taking into account that $[ad_J, \mathbf{c}(\psi)] = \mathbf{c}(J\psi)$ and $J^2\psi = -\psi$, we obtain

$$[\mathbf{c}(\psi), ad_J^2(\mathbf{d}) + \mathbf{d}] = -2ad_J(L_{J\psi}) + ad_J^2(L_\psi) ,$$

where $L_\psi: = [\mathbf{d}, \mathbf{c}(\psi)]$, and finally

$$\begin{aligned} [\mathbf{c}(\psi'), [\mathbf{c}(\psi), ad_J^2(\mathbf{d}) + \mathbf{d}]] &= -2[\mathbf{c}(\psi'), ad_J(L_{J\psi})] + [\mathbf{c}(\psi'), ad_J^2(L_\psi)] = \\ &= 2[\mathbf{c}(J\psi'), L_{J\psi}] - 2ad_J([\mathbf{c}(\psi'), L_{J\psi}]) - 2ad_J([\mathbf{c}(J\psi'), L_\psi]) - [\mathbf{c}(\psi'), L_\psi] + \\ &+ ad_J^2([\mathbf{c}(\psi'), L_\psi]) = 2\mathbf{c}(J([J\psi, \psi'] + [\psi, J\psi']) + [\psi, \psi'] - [J\psi, J\psi']) , \end{aligned}$$

which completes the proof.

□

As a corollary, the integrability condition (4.4) admits the following equivalent form:

$$[\mathbf{c}(J), [\mathbf{c}(J), \mathbf{d}]] + \mathbf{d} = 0 . \quad (4.5)$$

Let us remark that, if we decompose the l.h.s. of (4.5) into the sum of homogeneous components with respect to the natural grading in $\Omega^*(M)$, the lowest degree term will give us $[\iota_\Pi, [\iota_\Pi, \mathbf{d}]] = 0$; this is equal to $\iota_{[\Pi, \Pi]} = 0$, where $[\Pi, \Pi]$ is the Schouten-Nijenhuis bracket of Π . Thus the vanishing of N_J implies, in particular, that Π is a Poisson bivector. The explicit derivation of the remaining homogeneous terms gives a complete set of compatibility conditions for I , Π , and ω (cf. [12] for the details).

Proposition 4.5

1. *If $\omega = 0$ then I is a complex structure and Π is a real part of a complex Poisson bivector which is holomorphic with respect I ;*
2. *If, on the other hand, $I = 0$ we reobtain an ordinary symplectic structure from a generalized complex one, such that Π then is the respective induced Poisson structure.*

Proof.

1. The algebraic conditions (4.2), imposed on J , will give us $I^2 = -1_{TM}$ and $I\Pi = \Pi I^*$. The first identity implies that I is an almost complex structure in the usual sense, therefore the complexified tangent and cotangent bundles admit the usual decomposition into the sum of $(1, 0)$ and $(0, 1)$ parts with respect to I :

$$T(M)^c = T^{(1,0)} \oplus T^{(0,1)}, \quad T^*(M)^c = T^{(1,0)*} \oplus T^{(0,1)*}.$$

The second commutation relation implies that the complexification of Π belongs to the direct sum of $Hom(T^{(1,0)*}, T^{(1,0)})$ and $Hom(T^{(0,1)*}, T^{(0,1)})$. Taking into account that Π is real, we get $\Pi = \Pi_h + \overline{\Pi}_h$, where $\Pi_h \in \Gamma(\Lambda^2 T^{(1,0)} M)$. The degree 0 homogeneous component of the integrability condition (4.5) asserts the vanishing of the Nijenhuis tensor of I , which means that I is a complex structure on M in the usual sense. The degree 1 homogeneous component of (4.5) gives us

$$[I^*, \delta_\Pi] + [d_I, \iota_\Pi] = 0, \quad (4.6)$$

where $\delta_\Pi = [d, \Pi]$ and $d_I = [d, I^*]$. The Hodge decomposition of differential forms with respect to I allows to decompose d into the sum of $(1, 0)$ and $(0, 1)$ parts, $\partial + \overline{\partial}$, and ι_Π into the sum of $(2, 0)$ and $(0, 2)$ parts (the contractions with $\overline{\Pi}_h$ and Π_h , correspondingly). Similarly, we have the decomposition of d_I : $d_I = i(\partial - \overline{\partial})$. Let us look at the components of the l.h.s. of (4.6) which are homogeneous with respect to the Hodge decomposition. The identity (4.6) gives us only two (dependent) conditions, the first of which is conjugated to the second one: $[\overline{\partial}, \iota_{\Pi_h}] = 0$ and $[\partial, \iota_{\overline{\Pi}_h}] = 0$. This holds if and only if Π_h is a holomorphic Poisson bivector.

2. Let $I = 0$, then the algebraic relations (4.2) gives us the only independent condition $\Pi\omega = -1_{TM}$, which means that ω is a non-degenerate 2-form and Π is the corresponding bivector field (up to a sign convention). As it was mentioned above, once J is integrable, Π is necessarily Poisson. Thus ω has to be a symplectic form. Conversely, provided ω is a symplectic form, we define Π such that $\Pi\omega = -1_{TM}$. Now we need to check the integrability condition (4.5). This gives us the only identity to verify: $[\delta_\Pi, \omega \wedge] + d = 0$. Since M is a symplectic manifold, its dimension is even, say $\dim M = 2m$. It is easy to check that $[\iota_\Pi, \omega \wedge] = \bar{n} - m$, where \bar{n} is an operator which counts the degree of a differential form and m is simply the multiplication on m . The identity $d\omega = 0$ is obviously equivalent to the commutator relation $[d, \omega \wedge] = 0$. Therefore

$$[\delta_\Pi, \omega \wedge] + d = [[\iota_\Pi, d], \omega \wedge] = -[d, [\iota_\Pi, \omega \wedge]] + d = -[d, \bar{n} - m] + d = -d + d = 0.$$

□

5 Algebroids as Q-manifolds

Any Lie algebra \mathfrak{g} gives rise to a complex $(\Lambda^* \mathfrak{g}^*, d_{CE})$, where d_{CE} denotes the Chevalley-Eilenberg differential, $d_{CE}\alpha(\xi, \xi') = -\alpha([\xi, \xi'])$ for $\alpha \in \mathfrak{g}^*$ (and $\xi, \xi' \in \mathfrak{g}$) etc. By definition

an element $\alpha \in \mathfrak{g}^*$ is a linear function on \mathfrak{g} . Correspondingly, we may identify $\Lambda \mathfrak{g}^*$ with the space of functions on \mathfrak{g} if we declare the multiplication of two elements $\alpha, \alpha' \in \mathfrak{g}^*$, regarded as two functions, to be anticommuting, $\alpha\alpha' = -\alpha'\alpha$. This modified law of pointwise multiplication of functions is denoted by an additional Π (indicating parity reversion), $\Lambda \mathfrak{g}^* \cong C^\infty(\Pi\mathfrak{g})$. In fact, in what follows it will be important to not only consider a \mathbb{Z}_2 -grading, but a \mathbb{Z} -grading, inducing the \mathbb{Z}_2 -grading. Thus we declare elements of \mathfrak{g}^* to have degree 1, and then

$$\Lambda \mathfrak{g}^* \cong C^\infty(\mathfrak{g}[1]),$$

the bracket indicating that the canonical degree of the before mentioned vector space (which in the case of \mathfrak{g} is zero) is shifted by minus one. Given a basis e_a of elements in \mathfrak{g} , its dual basis θ^a becomes a set of coordinates on $\mathfrak{g}[1]$, any homogeneous element $\omega \in C^\infty(\mathfrak{g}[1])$ can be written as $\omega = \frac{1}{p!} \omega_{a_1 \dots a_p} \theta^{a_1} \dots \theta^{a_p}$, and the Chevalley-Eilenberg differential becomes a vector field of degree +1 (it raises the degree of homogeneity of any element by one), which we will denote by Q ,

$$Q = -\frac{1}{2} \theta^a \theta^b C_{ab}^c \frac{\partial}{\partial \theta^c}, \quad (5.1)$$

where C_{ab}^c denote the structure constants, $[e_a, e_b] = C_{ab}^c e_c$. By construction, $Q^2 = 0$, i.e. the vector field Q is *homological*.

This construction generalizes to the case where the Lie algebra acts on a manifold M by vector fields, $\xi \mapsto \rho(\xi) \in \Gamma(TM)$. Denoting $\rho(e_a)$ by ρ_a , the following vector field

$$Q = \theta^a \rho_a - \frac{1}{2} \theta^a \theta^b C_{ab}^c \frac{\partial}{\partial \theta^c}, \quad (5.2)$$

on $M \times \mathfrak{g}[1]$ is again homological. The corresponding cohomology in degree zero (functions containing no θ s) is obviously isomorphic to the space of functions on M invariant under the flow generated by the Lie algebra \mathfrak{g} . (5.2) and its cohomology may be viewed as the “BRST description” of the space of “gauge invariant” functions on M .

In the above ρ_a was a vector field on M , so in some local coordinates x^i of M one has $\rho_a = \rho_a^i \partial_i$, where ρ_a^i are (local) functions on M (and $\partial_i \equiv \partial/\partial x^i$). We may now consider to drop the restriction that C_{ab}^c is constant in (5.2) but instead also a function on M and pose the question under what conditions on ρ_a^i and C_{ab}^c the corresponding vector field (5.2) squares to zero. In this context Q is a vector field on a *graded manifold* \mathcal{M} ¹³ where the structure sheaf has local generators x^i and θ^a of degree zero and one, respectively; in this case we call \mathcal{M} of degree one. Note that since transition functions are by definition required to be degree preserving, a change of chart in \mathcal{M} requires $\tilde{\theta}^a = M_b^a \theta^b$, with M_b^a a local function on the *body* M of \mathcal{M} ; correspondingly, a graded manifold of degree one is isomorphic to a vector bundle $E \rightarrow M$, θ_a corresponding to a frame of local sections. Indicating that fiber-linear coordinates on E have degree one in the superlanguage, one writes $\mathcal{M} \cong E[1]$ in this case.

¹³In this article graded manifolds will always signify \mathbb{Z} -graded manifolds or even their special case of \mathbb{N}_0 -graded manifolds (cf. also below in the text). The \mathbb{Z} -grading induces naturally a \mathbb{Z}_2 -grading, which governs the signs of the algebra of “functions” on the graded manifold. By the forgetful functor a \mathbb{Z} -graded manifold thus also becomes a particular supermanifold.

A general graded manifold is equipped with an Euler vector field ϵ , such that the grading of the space of functions $\mathcal{F}(\mathcal{M})$ corresponds to the eigenvalue-decomposition of ϵ :

$$\mathcal{F}^k(\mathcal{M}) = \{f \in \mathcal{F}(\mathcal{M}) \mid \epsilon f = kf\}. \quad (5.3)$$

One may consider $k \in \mathbb{Z}$, but we will restrict ourselves to non-negative integers k except if explicitly stated otherwise.

Apparently, in the degree one case the extension of the algebra of functions on the body, $C^\infty(M) \cong \mathcal{F}^0(\mathcal{M})$, to the algebra of functions $\mathcal{F}(\mathcal{M})$ is generated by $\mathcal{F}^1(\mathcal{M})$, which can be thought of as the space of sections of a vector bundle (since $\mathcal{F}^1(\mathcal{M})$ has to be a locally free module over $\mathcal{F}^0(\mathcal{M})$). If such a graded manifold is equipped with a degree plus one homological vector field Q , the most general ansatz of which has the form of (5.2) with now C_{bc}^a being permitted to be functions on M , i.e. if one considers what is called a *Q-manifold of degree one* [37], the vector bundle $E \rightarrow M$ becomes equipped with the structure of a Lie algebroid: This may be regarded as a reformulation of Prop. 2.7 by the simple identification $C^\infty(\mathcal{M}) \cong \Omega_E(M)$. Alternatively, one may check directly that $Q^2 = 0$ implies in homogeneity degree two the morphism property of ρ , Lemma 2.5, and in degree three the Jacobi condition (2.1), both expressed for the local frame e_a . Note that in this picture the Leibniz rule (1.4) follows only from a change of coordinates on \mathcal{M} , requiring $Q^2 = 0$ to be valid in all possible frames; on the other hand, the bracket becomes automatically antisymmetric when defined by means of $[e_a, e_b] := C_{ab}^c e_c$, again to hold in any frame (which turns out to be consistent with the Leibniz rule (1.4)).¹⁴

If in a Lie algebroid there exists a frame such that the homological vector field takes the form with *constant* structure functions C_{ab}^c , then this algebroid is called an *action Lie algebroid*. The bundle E is then isomorphic to $M \times \mathfrak{g}$ for some Lie algebra \mathfrak{g} acting on M . Certainly, in general a Lie algebroid is not of this form and there does not exist a frame, not even locally, such that the structure functions would become constants.

On the other hand one may consider a graded manifold \mathcal{M} that carries a *homogeneous* symplectic form ω of degree n , i.e.

$$L_\epsilon \omega = n\omega, \quad (5.4)$$

in which case \mathcal{M} is called a *P-manifold* of the degree n . Note that the non-degeneracy of the symplectic form then requires that also \mathcal{M} has degree n (as mentioned, we do not consider graded manifolds with generators of negative degrees here, in which case this statement would no more be true). In the case of $n = 1$, we already found that \mathcal{M} is canonically isomorphic to $E[1]$, E a vector bundle over M . It is now easy to see that the P-structure restricts this further [32], $\mathcal{M} \cong T^*[1]M$, $(x^i, \theta^a) \cong (x^i, p_i)$, equipped with the canonical symplectic form $\omega = dx^i \wedge dp_i$. Indeed, since the symplectic form is of degree $n = 1$, eq. (5.4) implies $\omega = d\alpha$ with $\alpha = \iota_\epsilon \omega$. Suppose, (x^i, θ^a) are local coordinates of degree 0 and 1, respectively. Taking into account that ω is of degree 1, we immediately obtain that the expression of the symplectic form cannot contain $d\theta^a \wedge d\theta^b$, and since $\epsilon = \theta^a \partial / \partial \theta^a$, α it has to be of the form: $\alpha = \alpha_i^j(x) \theta_j dx^i$. α provides a morphism

¹⁴For a detailed discussion of such type of arguments, which, at least in a slightly more general context, turn out to be more tricky than one may expect at first sight, cf. [18, 17].

$TM \rightarrow E^*$; nondegeneracy of ω requires that this is an isomorphism, and thus (M, ω) is isomorphic to $T^*[1]M$ together with the canonical symplectic form. Note that functions on $T^*[1]M$ may be identified with multivector fields. The odd Poisson bracket induced by ω is then easily identified with the Schouten-Nijenhuis bracket.

A *PQ-manifold* of degree $n \in \mathbb{N}_0$ is then simultaneously a Q and a P manifold of degree n , such that Q preserves the symplectic form ω , in which case it turns out to be even Hamiltonian (cf. Lemma 2.2 in [32]): $Q = \{Q, \cdot\}$ for some function Q of degree $n+1$ (since the Poisson bracket decreases the degree by n), $Q^2 = 0$ reducing to $\{Q, Q\} = 0$.

In general, we use the following sign conventions: the algebra of differential forms on a graded manifold M is defined as $C^\infty(T[1]M)$. Then the degree of dh is $|h| + 1$, where $|h|$ is the degree of $h \in C^\infty(M)$. The Hamiltonian function of a Hamiltonian vector field is obtained from the relation $\iota_{X_h}\omega = (-1)^{|h|+1}dh$. The advantage of such a sign convention is that, if we produce a Poisson bracket by the formula $\{f, h\} = X_f(h)$, then the Lie algebra morphism property will hold: $[X_f, X_h] = X_{\{f, h\}}$. For a symplectic structure ω of degree n written in local Darboux coordinates $\omega = \sum_\alpha dp_\alpha \wedge dq^\alpha$ the local Poisson bracket follow to be $\{p_\alpha, q^\beta\} = (-1)^{n|q^\alpha|}\delta_\alpha^\beta$.

Now it is easy to see that a PQ-manifold of degree one is in one-to-one correspondence with a Poisson manifold M : $M \cong T^*[1]M$ and a degree two function has the form $Q \equiv \Pi = \frac{1}{2}\Pi^{ij}p_i p_j$, corresponding to a bivector field on M . The condition that the Schouten-Nijenhuis bracket of Π with itself vanishes, $\{Q, Q\} = 0$, is just one way of expressing the Jacobi condition for the Poisson bracket of functions (cf. also examples 2.3 and 3.3). The usual Poisson bracket $\{\cdot, \cdot\}_M$ between functions on M is reobtained here as a derived bracket: $\{f, g\}_M = \{\{f, -\Pi\}, g\}$ for any $f, g \in C^\infty(M) \subset C^\infty(M)$, since the right hand side indeed yields $\Pi^{ij}f_{,i}g_{,j}$ in local coordinates.

We quote a likewise result for the degree two case without proof, which is due to Roytenberg [32]:

Theorem 5.1 *A P-manifold of degree two is in one-to-one correspondence with a pseudo-Euclidean vector bundle $(E, {}^E g)$. A PQ-manifold of degree two is in one-to-one correspondence with a Courant algebroid $(E, {}^E g, \rho, [\cdot, \cdot])$.*

We add some remarks on this theorem for illustration. In appropriate Darboux-like coordinates on M , the symplectic form reads

$$\omega = dx^i \wedge dp_i + \frac{1}{2}\kappa_{ab}d\theta^a \wedge d\theta^b \quad (5.5)$$

where (x^i, θ^a, p_i) are coordinates of degree zero, one, and two, respectively, the vector bundle E corresponds to the graded submanifold spanned by (x^i, θ^a) , and the constants κ_{ab} correspond to the fiber metric ${}^E g$ evaluated in some orthonormal frame θ_a .

The symplectic form is of degree two, so Q is necessarily of degree three and thus of the form

$$Q = \rho_a^i \theta^a p_i - \frac{1}{6}C_{abc}\theta^a \theta^b \theta^c, \quad (5.6)$$

with coefficient functions ρ_a^i and C_{abc} depending on x^i only. Sections ψ of E may be identified with functions of degree one on \mathcal{M} , $\psi = \psi^a \theta_a$, $\theta_a \equiv \kappa_{ab} \theta^b$, functions f on M with functions of degree zero on \mathcal{M} . The anchor ρ and the Courant bracket $[\cdot, \cdot]$ now follow as derived brackets,

$$\rho(\psi)f = \{\{\psi, Q\}, f\} \quad (5.7)$$

$$[\psi, \psi'] = \{\{\psi, Q\}, \psi'\}, \quad (5.8)$$

while the fiber metric comes from the normal Poisson bracket: $Eg(\psi, \psi') = \{\psi, \psi'\}$. From (5.7) one obtains in particular $\rho(\theta_a)x^i = \rho_a^i$ and $[\theta_a, \theta_b] = C_{ab}^c \theta_c$, where the last index in C_{abc} has been raised by means of the fiber metric and θ_a is regarded as a local (orthonormal) frame in E .

Note that a derived bracket is in general not antisymmetric; from (5.8) one concludes $[\psi, \psi] = \frac{1}{2}\{\{\psi, \psi\}, Q\} = \frac{1}{2}\theta^a \rho_a^i \partial_i Eg(\psi, \psi)$, which reproduces the last axiom in definition 2.8. On the other hand, evaluated in an orthonormal frame θ_a the bracket does become antisymmetric, $[\theta_a, \theta_b] = C_{ab}^c \theta_c$.¹⁵ It is also obvious from (5.7) and (5.8) that the Courant bracket satisfies the Leibniz property in the second entry of the bracket (while it does not in the first one due to the non-antisymmetry).

The study of higher degree PQ manifolds is certainly more involved. They, however, always give rise to Loday algebroids in the following way

Proposition 5.2 *Given a PQ-manifold \mathcal{M} of degree $n > 1$ the functions ψ, ψ', \dots of degree $n - 1$ can be identified with sections in a vector bundle E . The formulas (5.7) and (5.8), where Q denotes the Hamiltonian for the Q -structure, equip E with the structure of a Loday algebroid.*

Proof. First we note that the functions of any fixed degree d on an \mathbb{N}_0 -graded manifold are a *locally free module* over the functions of degree zero, which in turn are isomorphic to $C^\infty(M)$. This implies the existence of vector bundles for any of those degrees d over M . For $d = n - 1$ we call this bundle E .

As remarked above, the vector field Q compatible with the symplectic form is always Hamiltonian (even for $n \in \mathbb{N}$); since the (graded) canonical Poisson bracket has degree $-n$, the respective Hamiltonian Q has to have degree $n + 1$. By an elementary computation adding up respective degrees, one then finds that $\rho(\psi)$ indeed maps functions $f \in C^\infty(M)$ to functions and the bracket $[\cdot, \cdot]$ takes again values in the sections of E .¹⁶ It is also obvious that $\rho(\psi)$ is a vector field on M , since the r.h.s. of the defining expression (5.7) satisfies an (ungraded) Leibniz rule for f being a product of two functions. To have an algebroid (cf. our definition 1.1), we need to verify two things: first, the Leibniz property (1.4), which follows at once from the two defining expressions on E above and the graded

¹⁵The difference to the situation with C_{ab}^c in the vector field Q of a Lie algebroid is that there such an equation holds in all frames and also that here a change in frame results in a different transformation property of C_{abc} in (5.6) by lifting this transformation to a canonical one on \mathcal{M} (which prescribes a particular induced transformation property for p_i). Cf. also [18] for many more details on this issue.

¹⁶We denote the expressions on the graded manifold and those isomorphic to them on $E \rightarrow M$ by the same symbols.

Leibniz property of the Poisson bracket $\{\cdot, \cdot\}$. Second, that $\rho(\psi)$ is indeed $C^\infty(M)$ -linear in ψ so as to really give rise to a bundle map $\rho: E \rightarrow TM$. The only potentially dangerous term which may violate this condition may arise when θ has terms quadratic or higher in the momenta p_i conjugate to the coordinates x^i on M . However, since the coordinates p_i necessarily have degree n and Q has degree $n + 1$, this is not possible for $n > 1$.

We are left with verifying the Loday property (2.1). It is only here where the condition $\{Q, Q\} = 0$, resulting from $Q^2 = 0$, comes into the game. We leave the respective calculation, which also makes use of the graded Jacobi identity of the Poisson bracket, as an exercise to the reader. \square

Some remarks: First of all, it is clear from the above proof that any symplectic graded manifold of degree $n \geq 2$ equipped with an arbitrary function Q of degree $n + 1$ gives, by the above construction, rise to an algebroid structure (in our sense, cf. Definition 1.1). If in addition $\{Q, Q\} = 0$, this algebroid becomes a Loday algebroid.

Certainly, the higher in degrees we go, the more additional structures arise. Already for the case of $n = 2$ the Loday algebroid had the additional structures making it into a Courant algebroid. For higher n , however, there will often be other algebroids out of which the Loday algebroid will be composed. Let us illustrate this for $n = 3$: In Darboux coordinates the symplectic structure will have the form

$$\omega = dx^i \wedge dp_i + d\theta^a \wedge d\xi_a, \quad (5.9)$$

where the coordinates x^i , θ^a , ξ_a , and p_i have the degrees 0, 1, 2, and 3, respectively. From this we learn that a PQ-manifold of degree 3 has to be symplectomorphic to $T^*[3](V[1])$ with its canonical symplectic form. Here $V \rightarrow M$ is a vector bundle and the brackets indicate shifts of degree in the respective fiber coordinates (note that without a shift the momenta ξ_a conjugate to the degree 1 fiber linear coordinates θ^a on $V[1]$ have to have degree -1 while they are now shifted so as to have degree 2). With $\psi = \varphi^a \xi_a + \frac{1}{2} \alpha_{ab} \theta^a \theta^b$, we see that the above vector bundle E in this case is isomorphic to $V \oplus \Lambda^2 V^*$. Now let us consider the Hamiltonian for the Q -structure, i.e. a function of degree four; it thus has to have the form

$$Q = \rho_a^i \theta^a - \frac{1}{2} C_{ab}^c \theta^a \theta^b \xi_c + \frac{1}{2} \beta^{ab} \xi_a \xi_b + \frac{1}{24} \gamma_{abcd} \theta^a \theta^b \theta^c \theta^d. \quad (5.10)$$

Obviously, for $\beta = \gamma = 0$ this is nothing but the canonical lift of the Q -structure (5.2) corresponding to a Lie algebroid, thus equipping V with the structure of a Lie algebroid. In the general case V still is an almost Lie algebroid and E can be considered as an appropriate extension into a Loday algebroid. (V itself is *not* a Loday algebroid, having a Jacobiator controlled by a contraction of β and γ . Adding a V -2-form α to the section φ , on which the anchor ρ acts trivially, one can restore the Loday property (2.1).)

We know that for $n = 1$ we also obtained an algebroid, even a Lie algebroid. However, this was defined on T^*M . The above construction leads to a trivial \mathbb{R} -bundle over M instead, the sections of which can be identified with functions on M , and $\rho: C^\infty(M) \rightarrow \Gamma(TM)$ is \mathbb{R} -linear but no more $C^\infty(M)$ -linear, in fact it corresponds (up to a sign being subject of conventions) to the map from functions to their Hamiltonian vector fields. In fact, both equations (5.8) and (5.7) become equivalent in this degenerate case, equipping

$C^\infty(M)$ with the Poisson bracket $\{\cdot, \cdot\}_M$, as already remarked above. (This bracket defines a Lie algebra structure on the sections of $M \times \mathbb{R}$, but not a Lie algebroid or even a general algebroid structure on this bundle since one does not have an anchor map for it).

To obtain the Lie algebroid structure on T^*M from the (P)Q-manifold $T^*[1]M$ we need to proceed differently. In fact, this provides a procedure that is applicable for any Lie algebroid $E \rightarrow M$, corresponding to a degree one Q-manifold $E[1]$. Sections of E are in 1-1 correspondence with vector fields ψ, ψ', \dots of degree minus one on $E[1]$, and their derived bracket $[[\psi, Q], \psi']$, where the brackets denote the (super)commutator of vector fields, is rather easily verified to reproduce the Lie algebroid bracket between the respective sections.

This procedure can in fact be considered for a Q-manifold of any degree n . While for general n this leads to what one may call Vinogradov algebroids (Loday algebroids with additional structures, cf. [17]), for $n = 2$ one obtains a V -twisted Courant algebroid as defined in Sec. 2 above. The degree -1 vector fields define the sections of the bundle E , the degree -2 vector fields the sections of the other bundle V ; the derived bracket $[[\psi, Q], \psi']$ of the degree -1 vector fields defines the Loday bracket on E , the ordinary commutator bracket $[\psi, \psi']$, which apparently takes values in the degree -2 vector fields, gives the V -valued inner product on E . All the defining properties of a V -twisted Courant algebroid are rather easy to verify in this case.

On the other hand, a Q-manifold \mathcal{M} of degree 2 gives always rise to $V[2] \rightarrow \mathcal{M} \rightarrow W[1]$, where the first map is an embedding, setting the degree one coordinates in \mathcal{M} to zero and the second map a projection, forgetting about the degree two coordinates. (Here V is the same bundle as the one in the previous paragraph and W another vector bundle over the same base manifold). A somewhat lengthy analysis (cf. [17]) shows that after one has chosen an embedding of $W[1]$ into \mathcal{M} , which composed with the projection giving the identity map, the degree 2 Q-manifold is in bijection with a Lie 2-algebroid. The vector bundle E of the V -twisted Courant algebroid picture is then composed of W and V , similarly to the situation of the degree 3 PQ manifold discussed above; one finds easily that $E \cong W \oplus W^* \otimes V$, and, under only a few more assumptions (like that the rank of V is at least two) on a V -twisted Courant algebroid of this form, also vice versa, the latter arises always from a Lie 2-algebroid in such a way.

We finally remark that if \mathcal{M} is a Q-manifold of degree n , then $T^*[n]\mathcal{M}$ is a PQ-manifold of the same degree.¹⁷ E.g. $T^*[1]E[1]$, with $E[1]$ a Lie algebroid (and Q lifted canonically, certainly), is a degree 1 PQ-manifold, and thus isomorphic to the Lie algebroid of a Poisson manifold. Indeed, the dual bundle of a Lie algebroid is canonically a Poisson manifold,¹⁸ in accordance with the easy-to-verify isomorphism $T^*[1]E[1] \cong T^*[1]E$. So,

¹⁷T.S. is grateful to D. Roytenberg for this remark in the context of a talk on V -twisted Courant algebroids.

¹⁸The dual E^* of any Lie algebroid E becomes a Poisson manifold in the following manner: In order to define a Poisson bracket on E^* , it is obviously sufficient to do so on the fiber constant and fiber linear functions. The former are functions that arise as pullbacks from functions on the base manifold M , the latter are sections of E . It is then straightforward to verify that

$$\{f, f'\} = 0, \quad \{\psi, f\} = \rho(\psi)f, \quad \{\psi, \psi'\} = [\psi, \psi'], \quad (5.11)$$

valid for all $f, f' \in C^\infty(M)$, $\psi, \psi' \in \Gamma(E)$, defines a Poisson structure on E^* .

Poisson geometry on M can be viewed as a particular case of Lie algebroid geometry (considering T^*M), but also vice versa, a Lie algebroid structure on E as a particular (fiber-linear) Poisson structure (on E^*)—and likewise so in higher degrees: A Courant algebroid is a particular case of a V -twisted Courant algebroid and also of a Lie 2-algebroid, but, in an appropriate sense, also Lie 2-algebroids and their corresponding V -twisted Courant algebroids, can be viewed as particular Courant algebroids. However, as already the example of a Lie algebroid shows, this is not always the most convenient way of viewing them.

We conclude this section by returning to the question of morphisms of Lie algebroids and, more generally, of any algebroid described by a Q - or a PQ -manifold. A morphism of Q -manifolds is a degree preserving map $\varphi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ such that its pullback $\varphi^*: C^\infty(\mathcal{M}_2) \rightarrow C^\infty(\mathcal{M}_1)$ is a chain map, i.e. one has $Q_1 \circ \varphi^* = \varphi^* \circ Q_2$. It is a morphism of PQ -manifolds, if in addition it preserves the symplectic form, $\varphi^* \omega_2 = \omega_1$.

6 Sigma models in the AKSZ-scheme

In this section we want to discuss a particular class of topological sigma models that can be constructed in the context of algebroids. By topological we want to understand that the space of solutions to the classical field equations (the Euler Lagrange equations of the functional) modulo gauge transformations does not depend on structures defined on the base manifold Σ in addition to its topology and that for “reasonable” topology (the fundamental group of Σ having finite rank etc) it is finite dimensional.

In the context of ordinary gauge theories, one such a wellknown space is the moduli space of flat connections on Σ . A functional producing such a moduli space is the Chern Simons theory

$$S_{CS}[A] = \frac{1}{2} \int_{\Sigma} \kappa \left(A \wedge (dA + \frac{1}{3}[A \wedge A]) \right), \quad (6.1)$$

defined on the space of connections of a trivialized G -bundle over an orientable three-dimensional base manifold Σ when specifying κ such that $(\mathfrak{g} = \text{Lie}(G), [\cdot, \cdot], \kappa)$ gives a quadratic Lie algebra; such a connection is represented by a \mathfrak{g} -valued 1-form A on Σ , $A \in \Omega^1(\Sigma, \mathfrak{g})$.

To find an appropriate generalization of this theory to the present context, let us first reinterpret the fields A and the field equations $F = 0$ of this model within the present context such that it permits a straightforward generalization. First of all, a \mathfrak{g} -valued 1-form A on Σ is evidently equivalent to a degree preserving map (a morphism)

$$a: T[1]\Sigma \rightarrow \mathfrak{g}[1]. \quad (6.2)$$

If e_a denotes a basis of \mathfrak{g} and θ^a the linear odd coordinates on $\mathfrak{g}[1]$ corresponding to a dual basis, then the 1-forms $A^a \equiv A_\mu^a \vartheta^\mu$ in $A = A^a \otimes e_a$ are given by the pullback of θ^a with respect to the map a , $A^a = a^*(\theta^a)$; here $\vartheta^\mu = d\sigma^\mu$ are the degree one coordinates on $T[1]\Sigma$ induced by (local) coordinates σ^μ on Σ . This is easy to generalize, given the background

of the previous sections: As our generalized gauge fields we will consider morphisms of graded manifolds

$$a: \mathcal{M}_1 \rightarrow \mathcal{M}_2 \quad (6.3)$$

keeping $\mathcal{M}_1 = T[1]\Sigma$ (so these gauge fields will be a collection of differential forms on Σ of various form degrees) and taking \mathcal{M}_2 as a more general \mathbb{N}_0 -graded manifold than $\mathfrak{g}[1]$.

In the example, not only $\mathcal{M}_1 = T[1]\Sigma$ but also $\mathcal{M}_2 = \mathfrak{g}[1]$ is not any graded manifold, but even a Q-manifold, Q_1 is the de Rham differential and Q_2 the Chevalley-Eilenberg differential (5.1). Let us reconsider the field equations $F = 0$ with $F = F^a \otimes e_a$ the curvature (1.6) of A in this context. Obviously $F^a = 0$, iff $dA^a = -C_{bc}^a A^b \wedge A^c$. With the identification of the vector fields Q_1 and Q_2 above, this in turn is now seen to be $Q_1 a^*(\theta^a) = a^* Q_2(\theta^a)$. In other words, the map $a^*: C^\infty(\mathcal{M}_2) \rightarrow C^\infty(\mathcal{M}_1)$ needs to be a chain map, or, according to the definition of a Q-morphism, the field equations of the Chern Simons gauge theory express that (6.3) is not only a morphism of graded manifolds but even a Q-morphism (a morphism of differential graded manifolds).

We are thus searching a functional defined on morphisms (6.3) such that its Euler-Lagrange equations forces these to become morphisms

$$a: (\mathcal{M}_1, Q_1) \rightarrow (\mathcal{M}_2, Q_2). \quad (6.4)$$

Certainly, for a true generalization of the *gauge* theory defined by means of (6.1), we also need to reinterpret its gauge transformations appropriately, so that we can formulate also the desiderata for the gauge symmetries of the searched for action functional. It turns out that on the solutions of (6.4) the gauge symmetries receive the interpretation of Q-homotopy.¹⁹ We are thus searching for a functional defined on (6.3) such that the moduli space of classical solutions modulo gauge transformations is the space of Q-morphisms from $(T[1]\Sigma, d)$ to the target Q-manifold (\mathcal{M}_2, Q_2) modulo Q-homotopy.

A functional can be obtained by the so-called AKSZ-method [1] (cf. also [7] and [31]) for the case that the target carries also a compatible symplectic form, i.e. that the target is a PQ-manifold. However, this method yields in fact already the BV-extension of the searched-for (“classical”) functional. We thus briefly recall some basic ingredients of the BV-formalism and we will do this at the example of a toy model, where we consider functions instead of functionals, as well as for the Chern-Simons theory (6.1) above.

The toy model is the following one: Consider a function S_{cl} on a manifold M invariant with respect to the action of some Lie algebra \mathfrak{g} : $\nu(S_{cl})(x) = 0$ for all points $x \in M$ and all elements $\nu \in \Gamma(TM)$ corresponding to the action of an element of \mathfrak{g} . We learnt that $M \times \mathfrak{g}$ carries the structure of a Lie algebroid, the action Lie algebroid. Let us thus, more generally, consider a Lie algebroid $E \rightarrow M$ together with a function $S_{cl} \in C^\infty(M)$ constant along the Lie algebroid orbits on M , i.e. s.t. $\rho(\psi)S_{cl} = 0$ for all $\psi \in \Gamma(E)$.

$S_{cl}(x)$ is supposed to mimic a functional invariant w.r.t. some gauge transformations (in the more general sense, cf., e.g., [19], also for general details on the BV formalism). Consider for simplicity first the case of the action Lie algebroid again. The gauge invariance here corresponds to $\delta_\epsilon S_{cl} = 0$ where $\delta_\epsilon x^i = \epsilon^a \rho_a^i(x)$ with some arbitrary \mathfrak{g} -valued

¹⁹We refer to [4] for the details.

parameters $\epsilon = \epsilon^a e_a$ (and $\rho(e_a) = \rho_a^i(x) \frac{\partial}{\partial x^i}$). In the BRST–BV approach one first replaces the parameters ϵ^a by anticommuting variables θ^a so that the action of the BRST operator δ on the original, classical fields (coordinates here) becomes $\delta x^i = \theta^a \rho_a^i(x)$. This is completed by the action of δ on the “odd parameters” θ^a , $\delta \theta^a = -\frac{1}{2} C_{bc}^a \theta^b \theta^c$ rendering δ nilpotent, $\delta^2 = 0$. In fact, this BRST operator is evidently nothing but the operator Q defined on $E[1]$, $\delta \equiv Q$, cf. eq. (5.2), and this works for a general Lie algebroid E with

$$\rho(e_a) = \rho_a^i(x) \frac{\partial}{\partial x^i}, \quad [e_a, e_b] = C_{ab}^c(x) e_c, \quad (6.5)$$

where now e_a is a local frame of sections of E and C_{ab}^c thus became structure functions (instead of just structure constants).

To obtain the BV-form of the action, one now turns to the (graded) phase space version of this, i.e. one introduces momenta (shifted in degree, cf. below) for each of the fields (coordinates on $E[1]$ in our example), which, conventionally, are called the *antifields*. In our example x_i^* conjugate to x^i and θ_a^* conjugate to θ^a . Adding the Hamiltonian lift of δ to the classical action we obtain

$$S_{BV} = S_{cl}(x) + \theta^a \rho_a^i(x) x_i^* - \frac{1}{2} C_{ab}^c(x) \theta^a \theta^b \theta_c^*. \quad (6.6)$$

To have this to have a uniform total degree, we need to shift the momenta in degree by minus one, $\deg x_i^* = -1$, $\deg \theta_a^* = -2$. Thus the BV odd phase space we are looking at together with the BV-function are of the form

$$\mathcal{M}_{BV} = T^*[-1]E[1], \quad \omega_{BV} = dx_i^* \wedge dx^i + d\theta_a^* \wedge d\theta^a, \quad S_{BV} = S_{cl} + Q, \quad (6.7)$$

where Q is the Hamiltonian of the canonical Hamiltonian lift of the vector field Q of the Lie algebroid $E[1]$, the odd Poisson bracket, the so-called BV-bracket $\{\cdot, \cdot\}_{BV}$ has degree +1, so that $Q_{BV} = \{S_{BV}, \cdot\}_{BV}$ has degree +1 as well. Q_{BV} is a differential, i.e. S_{BV} satisfies the so-called *classical master equation*

$$\{S_{BV}, S_{BV}\}_{BV} = 0, \quad (6.8)$$

which is completely obvious from our perspective: Q Poisson commutes with itself since it is the Hamiltonian for the Lie algebroid differential Q (there are no “odd constants”, $\{Q, Q\}_{BV}$ having degree 1), $S_{cl}(x)$ Poisson commutes with itself since it depends on coordinates only, and $\{S_{cl}(x), Q\} = 0$ since it corresponds to the Lie algebroid action on S_{cl} , which is zero by assumption.

So, $(\mathcal{M}_{BV}, \omega_{BV}, Q_{BV} = (S_{BV}, \cdot)_{BV})$ defines a PQ-manifold. In contrast to the previous \mathbb{Z} -graded Q-manifolds, this PQ-manifold also has negative degree generators. In fact, it is a cotangent bundle of an \mathbb{N}_0 -graded Q-manifold with a shift in the cotangent coordinate degrees such that it is precisely the momenta (antifields) that have negative degrees. The total degree is called the *ghost number* conventionally; so the classical fields (coordinates x in the toy model) have ghost number zero, the “odd gauge parameters” θ^a , the ghosts, have ghost number one, and the antifields have negative ghost number. We can also just consider the number of momenta or antifields: denoting this number by a subscript, we see that $S_{BV} = S_0 + S_1$ here, where $S_0 = S_{cl}$ and $S_1 = Q$.

In general, the BV formalism is more involved, there can be terms of higher subscript. Still, always S_0 is the classical action S_{cl} . Also we see that $\{x_i^*, S_{BV}\}_{BV}|_0 = \frac{\partial S_{cl}}{\partial x^i}$. Applying Q_{BV} to the classical fields and setting the antifield-less part to zero yields the critical points of S_{cl} , i.e. the classical field equations.

We now turn to the BV formulation of the Chern-Simons gauge theory (6.1). As before in (6.6), we add to the classical action (6.1) a term linear in the classical antifields A_a^* with a coefficient that is the (infinitesimal) gauge transformations, replacing the gauge parameters by odd fields β^a (so, naturally, A_a^* should be a 2-form on Σ), and we complete the expression by a term proportional to the odd (anti)fields β_a^* , 3-form on Σ , containing the structure constants such that the master equation (6.8) is satisfied;²⁰ this yields

$$S_{CS-BV}[A, A^*, \beta, \beta^*] = S_{CS}[A] + \int_{\Sigma} (d\beta^a + C_{bc}^a A^b \beta^c) A_a^* + \frac{1}{2} C_{ab}^c \beta^a \beta^b \beta_c^* \quad (6.9)$$

which is in a striking similarity with our toy model (6.6). Remembering that we could rewrite the toy model in a much more elegant form using the Q-language, cf. (6.7), we strive for a similar simplification in the present context.

For this purpose we first recall that the quadratic Lie algebra used to define the Chern-Simons theory is a Courant algebroid over a point, which in turn is a degree 2 PQ-manifold over a point (cf. Theorem 5.1):

$$\mathcal{M}_2 = \mathfrak{g}[1], \quad \omega = \frac{1}{2} \kappa_{ab} d\theta^a \wedge d\theta^b, \quad Q = -\frac{1}{6} C_{abc} \theta^a \theta^b \theta^c. \quad (6.10)$$

This is the target of the map (6.3), with the source, $\mathcal{M}_1 = T[1]\Sigma$, being a Q-manifold ($Q_1 = d$). The map (6.3) corresponds to the classical fields A , which we amended with further fields β, β^*, A^* above.

It is tempting to collect all these fields together into a super field (indices were raised by means of κ)

$$\mathcal{A}^a = \beta^a + A^a + A_*^a + \beta_*^a \quad (6.11)$$

by adding them up with increasing form degrees. In fact, this corresponds to an extension of the morphism (6.3) to what is called a *map* \tilde{a} from \mathcal{M}_1 to \mathcal{M}_2 . But before commenting on this extension on this more abstract level, we want to first convince ourselves that the concrete expression (6.11) is useful. Let us consider the action (6.1) simply replacing A by \mathcal{A} —and clearly keeping only the top degree forms for the integration over Σ . Viewing differential forms on Σ as graded functions on $T[1]\Sigma$, we can also write this as a Berezin integral over that graded manifold and we will partially do so in what follows. Let us consider the first part of this action first:

$$S_{source}[\mathcal{A}] = \frac{1}{2} \int_{T[1]\Sigma} \kappa(\mathcal{A} \wedge, d\mathcal{A}) = \frac{1}{2} \int_{\Sigma} \kappa(\beta, dA^*) + \kappa(A \wedge, dA) + \kappa(A^* \wedge, d\beta). \quad (6.12)$$

²⁰Again, also this example is a relatively simple one for what concerns the BV-formalism and the simpler BRST approach would be sufficient to yield the same results. However, already the models generalizing the Chern-Simons theory that we will discuss below, like the Poisson sigma model or the more general AKSZ sigma model, have a more intricate ghost and antifield structure.

We see that taking Σ to have no boundary for simplicity, with appropriate sign rules (see 6.20, in general), the first and the third term become identical and they reproduce all the terms containing a d in the BV-action above. Similarly,

$$S_{target}[\mathcal{A}] = \frac{1}{6} \int_{T[1]\Sigma} \kappa(\mathcal{A} \wedge [\mathcal{A} \wedge \mathcal{A}]) \equiv \frac{1}{6} \int_{T[1]\Sigma} C_{abc} \mathcal{A}^a \wedge \mathcal{A}^b \wedge \mathcal{A}^c \quad (6.13)$$

is easily seen to reproduce the remaining terms in (6.9). So we see that

$$S_{CS-BV}[A, A^*, \beta, \beta^*] = S_{CS}[\mathcal{A}] = S_{source}[\mathcal{A}] + S_{target}[\mathcal{A}] \quad (6.14)$$

and we will strive at understanding this action in analogy to the toy model situation (6.7).

For this purpose we need to identify the BV phase space of this situation. We thus first return to a further discussion of the superfields (6.11). The notion of a smooth map of graded manifolds is an extension of the notion of a morphism:²¹

$$\tilde{a} \in \text{Map}(\mathcal{M}_1, \mathcal{M}_2), \quad a \in \text{Mor}(\mathcal{M}_1, \mathcal{M}_2) = \text{Map}_0(\mathcal{M}_1, \mathcal{M}_2), \quad P_0 \tilde{a} = a. \quad (6.15)$$

In the case of a flat target manifold like $\mathfrak{g}[1]$ the description of a smooth map is rather clear from the example (6.11): we (formally) allow the functional dependence of the target coordinates on arbitrary degrees on the source. $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$ is naturally graded: The coefficients in the expansion like (6.11) are coordinates on this map space. Since the coordinates on the target $\mathcal{M}_2 = \mathfrak{g}[1]$ have degree one, each term in the expansion (6.11) has degree 1 as well. Correspondingly, the ghost β^a has degree 1, $A_\mu^a(\sigma)$, the coefficient in $A^a = A_\mu^a \vartheta^\mu$ has degree zero (since $\vartheta^\mu = d\sigma^\mu$, coordinates on the source $T[1]\Sigma$, have degree 1), $(A^a)_\mu{}^\nu(\sigma)$, the coefficients of the 2-form field A^a_* , are fields of degree -1 etc. A field (or antifield) is the same as a coordinate on $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$. Its degree zero part $\text{Map}_0(\mathcal{M}_1, \mathcal{M}_2)$ is the space of morphisms or the space of the original classical maps a , which correspond to the (classical) fields $A_\mu^a(\sigma)$. a results from \tilde{a} by projection (formal operator P_0 in (6.15)) to its degree zero part, which are those maps that are degree preserving: in the example (6.11) this is keeping the second term.

Generally, the space of maps between graded manifolds \mathcal{M}_1 and \mathcal{M}_2 , denoted as $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$, is uniquely determined by the functorial property (cf. eg. [13] or also [32]): for any graded manifold Z and a morphism $\psi : Z \times \mathcal{M}_1 \rightarrow \mathcal{M}_2$ there exists a morphism $\tilde{\psi} : Z \rightarrow \text{Map}(\mathcal{M}_1, \mathcal{M}_2)$ such that $\psi = \tilde{e}v \circ (\tilde{\psi} \times \text{Id})$, where the evaluation map $\tilde{e}v$ is (formally) defined in the obvious way:

$$\tilde{e}v : \text{Map}(\mathcal{M}_1, \mathcal{M}_2) \times \mathcal{M}_1 \rightarrow \mathcal{M}_2, \quad (\tilde{a}, \tilde{\sigma}) \mapsto \tilde{a}(\tilde{\sigma}). \quad (6.16)$$

Though the map \tilde{a} underlying (6.11) is not a morphism of graded manifolds (it does not even induce an ungraded morphism of the associative algebras of functions!), the evaluation map $\tilde{e}v$ is.

²¹Strictly speaking, the first and consequently the third formula do not make sense: $\text{Map}(\mathcal{M}_1, \mathcal{M}_2)$ turns out to be an infinite dimensional graded manifold and, as any graded manifolds, only its degree zero part, the body, contains points; a graded manifold, like a supermanifold, is not even a set. Still, it is useful to *think* like this; like everything in supergeometry, the real definitions are to be given algebraically on the dual level.

So the BV-phase space is $\mathcal{M}_{CS-BV} = \text{Map}(T[1]\Sigma, \mathfrak{g}[1])$, or, more generally, $\mathcal{M}_{BV} = \text{Map}(\mathcal{M}_1, \mathcal{M}_2)$. It is canonically an odd (infinite dimensional, weakly) symplectic manifold. The symplectic form ω_{BV} is induced by means of the one on the target, eq. (6.10):

$$\omega_{BV} = \int_{T[1]\Sigma} \widetilde{ev}^* \omega = \frac{1}{2} \int_{T[1]\Sigma} \kappa_{ab} \delta \mathcal{A}^a \wedge \delta \mathcal{A}^b, \quad (6.17)$$

where δ denotes the de Rahm differential on \mathcal{M}_{BV} (so as to clearly distinguish it from the *vector field* d on \mathcal{M}_1 !). Note that the pullback by means of the evaluation map (6.16) of the 2-form ω produces a (highly degenerate) 2-form on $\mathcal{M}_{BV} \times \mathcal{M}_1$, which, moreover, still has degree 2 since \widetilde{ev} is a morphism, as remarked above. The Berezin integration over $T[1]\Sigma$ then reduces the degree of the resulting differential form by 3 (since Σ is three-dimensional), so that ω_{BV} has degree -1, such as in our toy model (6.7). We could write out the right hand side of (6.17) similarly to (6.12); this then makes it clear that, after the integration, the resulting 2-form is indeed (weakly) non-degenerate. Let us stress at this point that certainly this construction would *not* work when sticking to the purely classical fields A : A likewise expression $\int_{\Sigma} \kappa(\delta A \wedge \delta A)$ would be nonzero only for a *two*-dimensional surface Σ —in fact, this then is the symplectic form of the classical phase space of the Chern Simons theory. So, also at this point the extension from $a \in \text{Mor}(\mathcal{M}_1, \mathcal{M}_2)$ to $\widetilde{a} \in \text{Map}(\mathcal{M}_1, \mathcal{M}_2) \equiv \mathcal{M}_{BV}$ is essential, because only the latter space is naturally symplectic, and indeed the symplectic form has degree -1 so that the BV-bracket will have degree +1, as it should be.

From the above the first steps in the generalization of the Chern-Simons BV action to a more general setting is clear: We will keep $(T[1]\Sigma, d)$ as our source Q -manifold (\mathcal{M}_1, Q_1) , with Σ having a dimension d different from three in general. As target we need to choose at least a symplectic graded manifold, but in fact, like in our guiding example (6.10), we will consider a PQ-manifold $(\mathcal{M}_2, \omega_2, Q_2)$, of degree n in general (for $n > 0$ we can also replace the symplectic vector field Q_2 by its generating Hamiltonian function \mathcal{Q}_2 of degree $n + 1$). Now $\int_{T[1]\Sigma} \widetilde{ev}^* \omega_2$ gives a degree -1 symplectic 2-form on $\mathcal{M}_{BV} = \text{Map}(\mathcal{M}_1, \mathcal{M}_2)$, iff $d = n + 1$ (since the Berezin intergration reduces the degree of the 2-form by d).

It remains to rewrite the action (6.9) or (6.14) in a form that will resemble somewhat the BV-function of the toymodel (6.7). In particular, according to our assumptions on source $\mathcal{M}_1 = T[1]\Sigma$ and target (6.10), we are having a vector field $Q_1 = d$ and $Q_2 = \{Q, \cdot\}$ on the source and the target, respectively. Both of them give rise to a vector field on $\mathcal{M}_{BV} = \text{Map}(\mathcal{M}_1, \mathcal{M}_2)$:²² Identifying the tangent space at $\widetilde{a} \in \mathcal{M}_{BV}$ with $\Gamma(\mathcal{M}_1, \widetilde{a}^* T\mathcal{M}_2)$, the two vector fields give rise to

$$\widetilde{a}_* Q_1 \quad \text{and} \quad Q_2 \circ \widetilde{a}, \quad (6.18)$$

respectively. These two vector fields on \mathcal{M}_{BV} , both of degree 1, are (graded) commuting, as acting on the right and the left of the map \widetilde{a} . Their difference

$$\widetilde{f} := \widetilde{a}_* Q_1 - Q_2 \circ \widetilde{a} \quad (6.19)$$

²²Details for the remaining part of the paragraph and the following one can be found in [1], [7], and [31].

is nothing but the BV Operator Q_{BV} . It turns out that both vector fields are Hamiltonian with respect to the symplectic form (6.17), with the Hamiltonians being given (6.12) and (6.13), respectively. In the spirit of (6.7), we can now also bring the BV action into the form

$$S_{BV-AKSZ} = \int_{T[1]\Sigma} \iota_d \widetilde{e} v^*(\alpha) + (-1)^d \widetilde{e} v^* Q, \quad (6.20)$$

which defines the general AKSZ sigma model. Here d is the dimension of Σ , in the example of the Chern-Simons theory thus $d = 3$ and Q is the Hamiltonian (6.10). In the first term α is a primitive of ω , $\omega = d\alpha$ and ι_d denotes the contraction with $Q_1 = d$. With Σ having no boundary, the action (6.20) is independent of the choice of α . If q^α denote Darboux coordinates of the target PQ-manifold, which, as mentioned, has degree $d - 1$, i.e. $\omega = \frac{1}{2} \omega_{\alpha\beta} dq^\alpha \wedge dq^\beta$ with ω_{ab} being constants, we can choose $\alpha = \frac{1}{2} \omega_{\alpha\beta} q^\alpha \wedge dq^\beta$. Let $\mathcal{A}^\alpha = \widetilde{a}^*(q^\alpha)$. Then we can (somewhat formally) rewrite the AKSZ action “evaluated” at \widetilde{a} more explicitly as

$$S_{BV-AKSZ}[\mathcal{A}] = \int_{T[1]\Sigma} \frac{1}{2} \omega_{\alpha\beta} \mathcal{A}^\alpha d\mathcal{A}^\beta + (-1)^d \widetilde{a}^* Q. \quad (6.21)$$

In this form it is very easy to see that we reproduce from this the Chern-Simons theory in its form (6.14) upon the choice (6.13) together with $d = 3$.

In general, the AKSZ sigma model is defined for a degree $d - 1$ PQ-manifold on a d dimensional base. The classical action results from the BV form of it simply by replacing \widetilde{a} by its degree zero part a , i.e. with $A^\alpha = a^* q^\alpha$

$$S_{AKSZ}[A] = \int_{T[1]\Sigma} \frac{1}{2} \omega_{\alpha\beta} A^\alpha dA^\beta + (-1)^d a^* Q. \quad (6.22)$$

Again, in the Chern-Simons case we easily find the classical action (6.1) reproduced.

We can, however, now also parametrize this sigma model more explicitly by means of the considerations in section 5 for the lowest dimensions of Σ : For $d = 2$, we need to regard degree 1 PQ-manifolds, which we had found to be always of the form

$$\mathcal{M}_2 = T^*[1]M \quad , \quad \omega = dp_i \wedge dx^i \quad , \quad Q = \frac{1}{2} \Pi^{ij} p_i p_j \quad (6.23)$$

with Π a Poisson bivector (cf. example 3.3); i.e. the target data are uniquely determined by a Poisson manifold (M, Π) . The most general AKSZ sigma model for $d = 2$ is thus seen to be the Poisson sigma model [36], [21]:

$$S_{PSM}[X^i, A_i] = \int_\Sigma A_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) A_i \wedge A_j \quad (6.24)$$

where $X^i = a^* x^i$ are 0-forms on Σ , $A_i = a^*(p_i)$ 1-forms, and we used a more standard notation of integration over the (orientable) base manifold Σ .

For $d = 3$ we see that we get a (topological) sigma model for any Courant algebroid, cf. Theorem 5.1, and the corresponding sigma model is easily specified by means of (5.5)

and (5.6). We call it the Courant sigma model. It was obtained first by Ikeda in [22] and later by Roytenberg more elegantly by the present method [31].

For $d = 4$ the geometrical setting of the target, a degree 3 PQ manifold, has not yet been worked out in detail or given any name. But again we can write down the explicit sigma model in this case using (5.9) and (5.10).

How we presented the AKSZ sigma model, its main purpose is to find a topological action functional such that its classical field equations are precisely Q-morphisms, (6.4). While this was proven for the Poisson sigma model explicitly in [4], the present formalism permits an elegant short and general proof. As the comparison of (6.22) with (6.21) shows, the difference between the classical action and its BV-extension is that we merely have to perform the replacement (6.4) to go from one to the other. This is a very specific feature of the present topological models, the BV extension is usually not that simple to obtain for a general gauge theory; here, however, it works like this as we saw above. The BV-AKSZ functional (6.21) becomes stationary precisely when (6.19) vanishes (since this is its Hamiltonian vector field and the symplectic form is non-degenerate). Correspondingly, the variation of (6.22) results into the same equation, but where \tilde{a} is replaced by a , so the Euler Lagrange equations are equivalent to the vanishing of

$$f: \mathcal{M}_1 \rightarrow T[1]\mathcal{M}_2 \quad , \quad f := a_* Q_1 - Q_2 \circ a \quad , \quad (6.25)$$

where we shifted the degree of the tangent bundle to the target so as to have f being degree preserving like a . This in turn is tantamount to the chain property of a^* (cf. also Lemma 8.1 below).

We finally remark that also Dirac structures and generalized complex structures can be formulated with profit into the language of super geometry (the former ones as particular Lagrangian Q-submanifolds in the degree 2 QP-manifold $T^*[2]T[1]M$), which in part can be used also to formulate particular sigma models for them within the present scheme not addressed in the present article (and different from those of the following section). We refer the reader for example to [16] and [8].

7 Sigma models related to Dirac structures

As we have mentioned before, a Poisson manifold gives a particular example of a Dirac structure, determined by the graph of the corresponding bivector in $TM \oplus T^*M$. Similar to the Poisson sigma model (PSM), the target space of which is a Poisson manifold, we now want to consider a topological sigma model associated to any Dirac subbundle of an exact Courant algebroid. This Dirac sigma model (DSM) [25] is supposed to be at least equivalent to the PSM for the special choice of a Dirac structure that is the graph of a Poisson bivector. Also, we want to continue pursuing our strategy that its classical field equations should be appropriate morphisms. In fact, in lack of a good notion of a morphism of a Dirac structure, we will content ourselves with a Q-morphism again, i.e. a Lie algebroid morphism in this case (since any Dirac structure is in particular a Lie algebroid structure). The model will be two-dimensional. As we saw in the previous

section that the most general two-dimensional model obtained by the AKSZ scheme is the PSM, the DSM does not result from this method, at least not by its direct application.

The target space of the Dirac sigma model is a manifold together with a Dirac structure D in an exact Courant algebroid twisted by a closed 3-form H . The space-time is a 2-dimensional surface Σ . We need also some auxiliary structures—a Riemannian metric g on M and a Lorentzian metric h on Σ . A classical field of the DSM is a bundle map $T\Sigma \rightarrow D$. First this corresponds to the base map $X: \Sigma \rightarrow M$, which corresponds to a collection of “scalar fields”. Taking into account that $D \subset TM \oplus T^*M$, we represent the remaining field content by a couple of sections $V \in \Omega^1(\Sigma, X^*TM)$, satisfying the constraint that this couple combines into a section of $T^*\Sigma \otimes X^*D$. Here $\Omega^p(N, E)$ is by definition $\Gamma(\Lambda^p T^*N \otimes E)$ for any smooth manifold N and vector bundle $E \rightarrow N$. The DSM action is the sum of two terms, one using the auxiliary geometrical structures and one that uses the topological structures only: $S_{DSM}[X, V, A] = S_{geom} + S_{top}$, where

$$S_{geom} : = \frac{\alpha}{2} \int_{\Sigma} \|dX - V\|^2, \quad (7.1)$$

$$S_{top} : = \int_{\Sigma} \left(\langle A \wedge, dX - \frac{1}{2}V \rangle \right) + \int_{N^3} H. \quad (7.2)$$

Here α is a real (non-vanishing, in general) constant²³, the absolute value in the first term corresponds to the canonical pairing by $h \otimes g$ on $T\Sigma \otimes TM$,²⁴ the brackets \langle, \rangle denote the pairing between TM and T^*M , and the last term in the topological part of the action is the integral of H over an arbitrary map $N^3 \rightarrow M$ for $\partial N^3 = \Sigma$ which extends $X: \Sigma \rightarrow M$ (here we assumed for simplicity that X is homotopically trivial²⁵, which is e.g. always the case if the second homotopy group of M is zero).

This sigma model generalizes the G/G Wess-Zumino-Witten (GWZW) [41, 15] and (H -twisted) Poisson sigma model, simultaneously. We first comment on the relation to the PSM. Let us first choose $H = 0$ (cf. Example 3.3) and $\alpha = 0$ (yielding $S_{geom} \equiv 0$); with D being the graph of a bivector Π , we have $V = X^*\Pi(A, \cdot)$ and comparison with (6.24) shows that we indeed have $S_{DSM} = S_{PSM}$ in that case. For H non-zero we get the twisted version of the Dirac structure of example 3.3 and by means of eq. (7.2) of the corresponding sigma model [23], respectively. (We will comment on non-zero α below). The GWZW model results from a special choice of the Dirac structure in an exact Courant algebroid on a quadratic Lie group $M = G$, H is the Cartan 3-form, g the biinvariant Riemannian metric on G , and $\alpha = 1$. In the description of a Dirac structure by an orthogonal operator $S \in \text{End}(TM)$ once a metric g has been fixed on M , cf. Cor. 3.9, the Dirac structure on G is the one given by the adjoint action $S = \text{Ad}_g$, where $g \in M$ is the respective base point.

²³The Lorentzian signature of h is chosen for simplicity. In the Riemannian version of the Dirac sigma model the coupling constant α has to be totally complex, that is, $\alpha \in i\mathbb{R}$.

²⁴More explicitly this expression was defined after formula (1.1).

²⁵For less a topologically less restrictive setting, one permits action functionals up to integer multiples of $2\pi\hbar$ —here we refer to the literature on Wess-Zumino terms, cf. [25] and references therein.

The GWZW model is in so far an important special case as it is not only a well known model in string theory (cf., e.g., [15]), but it is also *known* to be topological, despite the appearance of auxiliary structures needed to define it. Also, its Dirac structure can be shown to *not* be a graph, e.g. by employing the characteristic classes described in sec. 3.

Part of the topological nature for the general DSM can be already verified on the level of the classical field equations: Do they depend on the auxiliary structures like g or h ? Here we cite the following result from [25]:

Theorem 7.1 *Let $\alpha \neq 0$, then a field (X, V, A) is a solution of the equations of motion, if and only if the corresponding bundle map $T\Sigma \rightarrow D$ induces a Lie algebroid morphism.*

Since the notion of a Lie algebroid morphism does not depend on auxiliary structures as those mentioned above, we see that at least this condition is satisfied for nonvanishing α .

The proof of this theorem is somewhat lengthy,²⁶ so that we do not want to reproduce it here; instead we want to prove it for the simplest possible Dirac structure, $D = TM$, example 2.2 above, and refer for the general fact to [25].

We start by calculating the field equations of the sigma model; but for $D = TM$ and $H = 0$ (cf. example 2.2) the topological part of the action is identically zero. So, it remains to look at the variation of (7.1). Since V is an independent (unconstrained) field in this case, the variation of the quadartic term w.r.t. V yields

$$dX = V \tag{7.3}$$

while the X -variation vanishes on behalf of that equation. Mathematically, this equation is tantamount to saying that the vector bundle morphism $a: T\Sigma \rightarrow TM$ is the push forward of a map $X: \Sigma \rightarrow M$, $a = X_*$. We obtain the required statement in this special case by use of the following

Lemma 7.2 *A Lie algebroid morphism from the standard Lie algebroid over a manifold Σ to the standard Lie algebroid over a manifold M is the push forward of a smooth map $X: \Sigma \rightarrow M$.*

Proof. Recall from section 5 that the Lie algebra morphism above can be defined best by a degree preserving map $\bar{a}: T[1]\Sigma \rightarrow T[1]M$ such that $\bar{a}^*: C^\infty(T[1]M) \rightarrow C^\infty(T[1]\Sigma)$ commutes with the respective differentials characterizing the Lie algebroid, here being just the respective de Rahm differentials. Let us choose local coordinates $\sigma^\mu, \vartheta^\mu = d\sigma^\mu$ and $x^i, \theta^i = dx^i$, respectively. Then $X^i = \bar{a}^*x^i$ corresponds to the base map X of the Lemma. On the other hand $V^i := \bar{a}^*\theta^i = d\bar{a}^*x^i = dX^i$, where in the second equality we used that \bar{a}^* commutes with d . $V^i = dX^i$ or, equivalently, $V_\mu^i = X^i_{,\mu}$ is the searched-for equation. \square

From the example we also see that $\alpha \neq 0$ is a necessary condition for the theorem to hold. Were $\alpha = 0$ in that special case, there were *no* field equations and the vector bundle

²⁶One of the complications is that the fields A and V are not independent from one another in general and this has to be taken care of when performing the variations. One way of doing that is by using the orthogonal operator S mentioned in Cor. 3.9, which permits us to express these two fields by means of an independent $W \in \Omega^1(\Sigma, X^*TM)$ according to $A = (1 + S)W$ and $V = (1 - S)W$.

morphism $a: T\Sigma \rightarrow TM$ were unrestricted and in general not a Lie algebroid morphism. On the other hand, it is not difficult to see (but it certainly also follows from the theorem) that for the (possibly H -twisted) PSM the field equations *do not change* when adding the term S_{geom} with some non-vanishing α . We thus propose to consider S_{DSM} for non-vanishing α in general.²⁷

There are two more important issues which we need to at least mention in this context. First, if the DSM is to be associated to a Dirac structure on M and all other structures used in the definition of the functional are to be auxiliary, one needs to show that it is in the end the *cohomology class* of H only that enters the theory effectively. In [25] we proved

Proposition 7.3 *The DSM action transforms under a change of splitting (2.17) according to*

$$S_{DSM} \mapsto S_{DSM} + \int_{\Sigma} X^*(B)(dX - V \wedge dX - V). \quad (7.4)$$

Let $\alpha \neq 0$ and B be a "sufficiently small" 2-form, then there exists a change of variables $\bar{V} = V + \delta V$ and $\bar{A} = A + \delta A$, such that

$$S_{DSM}[X, \bar{V}, \bar{A}] = S_{DSM}[X, V, A] + \int_{\Sigma} X^*(B)(dX - V \wedge dX - V).$$

So, a change of the splitting can be compensated for by a (local) diffeomorphism on the space of fields.

Secondly, we did not yet touch the issue of the gauge symmetries, neither in the previous section on the AKSZ sigma models nor for the DSMs. Certainly the gauge symmetries are of utmost importance in topological field theories (without them we were never able to arrive at a finite dimensional moduli space of solutions for instance). While the BV-formalism produces them by means of the BV-operator for the AKSZ sigma models (although possibly in a coordinate dependent way, cf., e.g., [4] addressing this issue), they are less obvious to find for the DSM. In fact, here also all the *auxiliary structures do enter*, cf. [25] for the corresponding formulas. It is only *onshell*, i.e. on using the field equations (here only $dX = V$), that the gauge transformations obtain a nice geometrical interpretation: they turn out to become "Lie algebroid homotopies" (cf. also [4]) in this case. We do, however, not want to go into further details on this here; in the present article we decided to focus more on the field equations, reassuring the reader in words that the more intricate gauge symmetries fit nicely into the picture as well, completing it in an essential way.

²⁷In [25] it is conjectured that the theory with $\alpha = 0$ is (essentially) equivalent to the theory with $\alpha \neq 0$ in general. E.g. for $D = TM$, and $\alpha \neq 0$ the moduli space of solutions to the field equations (maps $X: \Sigma \rightarrow M$) *up to gauge symmetries* (which at least contain the homotopies of this map X) is zero dimensional, like the moduli space for the vanishing action $\alpha = 0$. It is argued that the geometrical part serves as a kind of regulator for the general theory, which also should permit localization techniques on the quantum level.

8 Yang-Mills type sigma models

In the Introduction we recapitulated the idea of sigma models: one wants to replace the flat target space \mathbb{R}^n of usually a collection of n scalar fields (functions on spacetime Σ) by some geometrical object, like a Riemannian manifold. We posed the question, if, in the context of gauge fields (1-forms on space time), we can replace in a likewise fashion the “flat” Lie algebra \mathbb{R}^n (or, more generally, \mathfrak{g}) by some nontrivially curved geometrical object. In fact, the Poisson sigma model (6.24), or more generally, the AKSZ sigma model (6.20), provides, in some sense, half a step into this direction: Let us consider the three dimensional case, where this is most evident. The quadratic Lie algebra \mathfrak{g} needed for the definition of the Chern Simons gauge theory can be generalized to a Courant algebroid, associated to which is the Courant sigma model [22, 31], which when specializing to the “flat case” \mathfrak{g} reproduces the Chern Simons theory. While these models realize in a geometrically nice way the right (classical) field content, the target algebroid being represented by a PQ-manifold and the Lie algebra valued 1-forms of a Yang-Mills theory being interpreted as and generalized to degree preserving maps from $T[1]\Sigma$ to the respective target, and also the gauge transformations are generalized in a reasonably looking way, there is, from the *physical* point of view, a major drawback or “flaw” of these theories: What mathematically is usually considered an advantage of a field theory, namely to be topological, in the context of physics rules out a theory for being feasible to describe the degrees of freedom we see realized in the interacting world around us.

Indeed, the space of flat connections, which are the field equations of the Chern Simons theory, modulo gauge transformations is (for, say, Σ without boundary and of finite genus) a finite dimensional space and this generalizes in a likewise fashion to the moduli space of solutions modulo gauge transformations for all the AKSZ models, where one considers the space of Q-morphisms modulo Q-homotopy. The moduli space needed to host a physical particle (like a photon, electron, Higgs, etc), on the other hand, is always infinite dimensional (like the space of harmonic functions on Σ for a Laplacian corresponding to a $(d-1,1)$ -signature metric). There is also another way of seeing that one has gone half way only: Such as we want that when in a sigma model for scalar fields the choice of a “flat background” (i.e., in that case, that the target is a flat Riemannian manifold \mathbb{R}^n) the action reduces to (1.2), we want that when in a Yang-Mills type sigma model the target is chosen to be a Lie algebra \mathbb{R}^n or, more generally, \mathfrak{g} , the gauge theory reduces to (1.5) and (1.7), respectively. So we will pose this condition, maybe adding that there should be a “comparable number” of gauge symmetries in the general model as in the special case.

In a similar way we may generalize this condition by extending it to also higher form gauge fields B^b , namely that for an appropriate flat choice of the target geometry one obtains from the (higher) Yang-Mills type sigma model

$$S[B^b] = \frac{1}{2} \int_{\Sigma} dB^b \wedge *dB^b. \quad (8.1)$$

For the case that B s are 2-forms, this will yield an action functional for nonabelian gerbes.

Before continuing we bring the standard Yang-Mills action (1.7) into a form closer to a topological model first. We consider a trivial bundle in what follows, in which case the

curvature or field strength is a \mathfrak{g} -valued 2-form. If d denotes the dimension of spacetime Σ , we introduce a \mathfrak{g}^* -valued $d-2$ form Λ in addition to the connection 1-forms A . With κ^{-1} denoting the scalar product on \mathfrak{g}^* induced by κ on \mathfrak{g} , then

$$S'_{YM}[A, \Lambda] := \int_{\Sigma} \langle \Lambda, F \rangle + \kappa^{-1}(\Lambda \wedge * \Lambda) \quad (8.2)$$

is easy to be seen as equivalent to (1.7) on the classical level (Euler Lagrange equations) after elimination of the auxiliary field Λ . (Equivalence on the quantum level follows from a Gaussian integration over the field Λ , as it enters the action quadratically only—up to an overall factor, which is irrelevant for the present considerations).

The first part of this action is topological, it is only the second term, breaking some of the symmetries of the topological one, that renders the theory physical. Let us first generalize the topological part, a so-called BF-theory, by an appropriate reformulation. In fact, we can obtain the BF-theory from the AKSZ-method by considering $\mathcal{M}_2 = T^*[d-1]\mathfrak{g}[1]$ as a target QP-manifold (as before d is the dimension of spacetime Σ). Indeed, it is canonically a symplectic manifold with the symplectic form being of degree $d-1$, as it was required to be in section 6, and the Q-vector field on $\mathfrak{g}[1]$, eq. (5.1), can always be lifted canonically to the cotangent bundle with the following Hamiltonian²⁸

$$Q = (-1)^d \frac{1}{2} C_{bc}^a \theta^b \theta^c p_a. \quad (8.3)$$

The AKSZ action (6.22) now just produces

$$S_{BF}[\Lambda, A] = \int_{\Sigma} \Lambda_a (dA^a + \frac{1}{2} C_{bc}^a A^b \wedge A^c) \quad (8.4)$$

where we used $A^a = a^*(\theta^a)$ and $\Lambda_a = a^*(p_a)$; and this is just equal to the first term in (8.2). It is now easy to find a generalization of this part of the action for a general (\mathbb{N}_0 -graded) Q-manifold $(\widetilde{\mathcal{M}}_2, \widetilde{Q}_2)$: Just consider $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ as a target QP-manifold with the canonical lift of \widetilde{Q}_2 to the Hamiltonian vector field Q_2 (with Hamiltonian $Q_2 = Q$).

Let us exemplify this at the example of a Lie algebroid, where $\widetilde{\mathcal{M}}_2 = E[1]$ and \widetilde{Q}_2 is given by means of formula (5.2) (with $\rho_a = \rho_a^i \partial / \partial x^i$) and $\rho_a^i(x)$, $C_{ab}^c(x)$ being the Lie algebroid structure functions in the local coordinates x^i, θ^a on $E[1]$). Now the same procedure yields the action functional for a “Lie algebroid BF-theory” [38, 39, 5]

$$S_{LABF}[\Lambda_i, \Lambda_a, X^i, A^a] = \int_{\Sigma} (\Lambda_i \wedge F^i + \Lambda_a \wedge F^a) \quad (8.5)$$

where Λ_i and Λ_a are $d-1$ forms and $d-2$ forms, respectively, and

$$F^i = dX^i - \rho_a^i(X) A^a \quad (8.6)$$

$$F^a = dA^a + \frac{1}{2} C_{bc}^a(X) A^b \wedge A^c \quad (8.7)$$

²⁸We use the following sign convention: $\{p_a, q^b\} = (-1)^{m|q^b|} \delta_a^b$, where m is the degree of a symplectic form.

are the generalizations of the YM-curvatures, which we prefer to call *field strengths* in the general case. We use the notation $A^\alpha = a^*(q^\alpha)$, where q^α are graded coordinates on $\widetilde{\mathcal{M}}_2$ and $\Lambda_\alpha = a^*(p_\alpha)$, the corresponding momenta (or anti-fields) on $\mathcal{M}_2 = T[1]\widetilde{\mathcal{M}}_2$. Note that the map $a: T[1]\Sigma \rightarrow \mathcal{M}_2$ is degree preserving, so that $X^i \equiv A^i = a^*(x^i)$ are 0-forms on Σ or scalar fields, $A^a = a^*(\theta^a)$ are 1-forms, and Λ_i and Λ_a are $d-1$ and $d-2$ forms, respectively.

We remark in parenthesis that although (8.5) is inherently coordinate and frame independent, the expression (8.7) is not. It is the splitting of the field strength(s) into two independent parts (8.6) and (8.7), which cannot be performed canonically. One way of curing this is by introducing an additional connection on the (target) Lie algebroid E , cf., e.g., [4, 39]. Better is to realize that the whole information is captured by the map (6.25), taking values in the *tangent* bundle over the target $\mathcal{M}_2 = T^*\widetilde{\mathcal{M}}_2$. We now turn to this perspective.

The map f defined in (6.25) covers the map $a: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ (it is also this fact that makes the difference in (6.25) well-defined, being the difference between two elements in the fiber, which is a vector space, over the same point)

$$\begin{array}{ccc} & T[1]\mathcal{M}_2 & \\ f \nearrow & \downarrow & \\ \mathcal{M}_1 & \xrightarrow{a} & \mathcal{M}_2 \end{array}$$

and measures the deviation of a to be a Q-morphism, in which case the following diagram would commute

$$\begin{array}{ccc} T[1]\mathcal{M}_1 & \xrightarrow{a_*} & T[1]\mathcal{M}_2 \\ Q_1 \uparrow & & \uparrow Q_2 \\ \mathcal{M}_1 & \xrightarrow{a} & \mathcal{M}_2 \end{array}$$

Denote by $x^A = (q^\alpha, p_\alpha)$ the graded coordinates on $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ and by dx^A the induced fiber-linear coordinates on $T[1]\mathcal{M}_2$. To use the map f in practise, the following lemma is helpful:

Lemma 8.1 *Let (x^A, dx^A) be local coordinates on $T[1]\mathcal{M}_2$, $a \in \text{Mor}(T[1]\Sigma, \mathcal{M}_2)$, and f the map defined in (6.25). Then*

$$f^*(x^A) = a^*(x^A) \quad f^*(dx^A) = (da^* - a^*Q_2)x^A. \quad (8.8)$$

Proof. The first part is evident from f covering a as remarked above already. The second part is rather straightforward and we show only one first step in the calculation. The vector field Q_2 of degree 1 is a *degree preserving* section of the tangent bundle over \mathcal{M}_2 , if the latter is shifted by 1 in degree, i.e., as here, one considers $T[1]\mathcal{M}_2$. The section maps a point with coordinates x^A to the point $(x^A, Q_2^A(x))$. Thus we find for the pull-back of dx^A by this section: $(Q_2)^*dx^A = Q_2^A(x)$, which, however, can also be rewritten as Q_2 applied to

x^A . The remaining similar steps we leave as an exercise to the reader. We just remark that we had used $Q_1 = dd$ and that certainly pull-backs go the reverse order as the respective map, cf the second diagram above. \square

We remark also that dx^A is considered as a *function* on $T[1]\mathcal{M}_2$ and is pulled back as such, *not* as a differential form on $T[1]\mathcal{M}_2$, while a function on $T[1]\mathcal{M}_2$ is, on the other hand, equally well a differential form on \mathcal{M}_2 . In fact, the so-understood pull back will *not* be a chain map w.r.t. the de Rahm differential (a vector field on $T[1]\mathcal{M}_2$), but only w.r.t. a modified vector field on $T[1]\mathcal{M}_2$, cf. Prop. 8.3 below.

We see that the field strengths F^i and F^a are nothing but the pull back of dx^i and $d\theta^a$ by f and thus find

Corollary 8.2 *The action functional (8.5) can be written more compactly as*

$$S_{LABF}[a] = \int_{\Sigma} f^*(p_{\alpha} dq^{\alpha}) \quad (8.9)$$

where $a: T[1]\Sigma \rightarrow \mathcal{M}_2 = T^*[d-1]E[1]$ and $p_{\alpha}dq^{\alpha}$ is the canonical 1-form on \mathcal{M}_2 .

Now we are in the position to generalize the Yang-Mills action functional (1.7) or better (8.2) to the Lie algebroid setting, i.e. to a situation where the structural Lie algebra \mathfrak{g} is replaced by a Lie algebroid E (cf. definition 2.1). A Lie algebroid Yang-Mills theory is defined as a functional on $a \in \text{Mor}(T[1]\Sigma, \mathcal{M}_2 = T^*[d-1]E[1])$ and is of the form [39]

$$S'_{LAYM}[a] = S_{LABF}[a] + \int_{\Sigma} {}^E\mathfrak{g}^{-1}(\Lambda_{(d-2)} \wedge * \Lambda_{(d-2)}) \quad (8.10)$$

where $\Lambda_{(d-2)} \in \Omega^{d-2}(\Sigma, X^*E)$ are the Lagrange multiplier fields of the 2 form field strengths only. Equivalently, eliminating precisely those Lagrange multipliers like in the transition from (8.2) to (1.7), we can regard the somewhat more explicit action functional

$$S_{LAYM}[A^a, X^i, \Lambda_i] = \int_{\Sigma} \Lambda_i \wedge F^i + \frac{1}{2} {}^E\mathfrak{g}(X)_{ab} F^a \wedge *F^b, \quad (8.11)$$

where F^i and F^a are the field strengths (8.6) and (8.7), respectively, detailed above. In a standard YM theory, the metric on the Lie algebra needs to be ad-invariant. Here we can ask that $(E, {}^E\mathfrak{g})$ define a maximally symmetric E -Riemannian space (cf. Def. 2.10) or that the fiber metric is invariant w.r.t. a representation (2.13) induced by any auxiliary connection on E . (For still less restrictive conditions cf. [29]).

This is now easy to generalize to higher form degrees of the gauge field. Let us consider a tower of gauge fields X^i, A^a, \dots, B^B , which are 0-forms, 1-forms etc, respectively, up to a highest degree p , so B^B being p -forms. The role of the structural Lie algebra will now be played by a degree p Q-manifold $(\widetilde{\mathcal{M}}_2, Q_2)$. The gauge fields are collected into a (degree preserving) map $a: T[1]\Sigma \rightarrow \widetilde{\mathcal{M}}_2$. This is extended to $a \in \text{Mor}(T[1]\Sigma, \mathcal{M}_2)$ with $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ the corresponding PQ-manifold with canonical 1-form $p_{\alpha}dq^{\alpha}$. We then consider [40]

$$S'_{higherYM}[a] = \int_{\Sigma} f^*(p_{\alpha} dq^{\alpha}) + \int_{\Sigma} {}^V\mathfrak{g}^{-1}(\Lambda_{(d-p-1)} \wedge * \Lambda_{(d-p-1)}) \quad (8.12)$$

where ${}^V\mathbf{g}$ is a fiber metric on the vector bundle $V \rightarrow M$ corresponding to the degree p variables on \mathcal{M}_2 (in fact, there is a canonical quotient of \mathcal{M}_2 yielding $V[p]$). Similarly to before we can also eliminate the lowest form degree Lagrange multipliers $\Lambda_{(d-p-1)}$ in S'_{higherYM} to obtain $S_{\text{higherYM}}[a]$ in which the highest form degree field strength is squared by means of the fiber metric ${}^V\mathbf{g}$.

There is always a condition on this fiber metric generalizing the ad-invariance in the standard YM case. Let us specify this condition in the case $p = 2$. We were mentioning in section 5 that a degree 2 Q-manifold corresponds to a Lie 2-algebroid, cf. Def. 2.13 and the ensuing text. In this case the condition to be placed on ${}^V\mathbf{g}$ is its invariance w.r.t. the E -connection ${}^E\nabla$ on V [40]:

$${}^E\nabla {}^V\mathbf{g} = 0. \quad (8.13)$$

For $p = 2$ this is a possible definition of an action functional for nonabelian gerbes.

In this contribution we never discussed the gauge transformations in any detail. In a gauge theory they are certainly of utmost importance and much further motivation for the theories discussed here come from looking at their gauge invariance also.

We make here only two general remarks on the gauge transformations: First, for a general algebroid type gauge theory the gauge symmetries have a “generic part” that comes from the structural algebroid (possibly including some particularly important geometrical ingredients from the target like a compatible symplectic form), but there can be also contributions to them that show all possible structural ingredients used in the definition of the action functional. An (impressive) example for the latter scenario is provided by the Dirac sigma model, cf. section 7 and [25]. Second, for the higher Yang-Mills action functionals proposed above, or also any other *physical* gauge theory proposed in this setting, the BV-formulation, which also captures elegantly the gauge symmetries of a theory, will *not* just consist in replacing the morphisms a by supermaps \tilde{a} as it was the case with the AKSZ type sigma models. In fact, the space $\text{Map}(T[1]\Sigma, \mathcal{M}_2)$ is too big: It contains ghosts and ghosts for ghosts etc for gauge symmetries of the BF-part of the action that are broken by the fiber metric part. It is tempting, however, to try to obtain the BV-formulation of the full physical model by a kind of supersymplectic reduction induced by the symmetry breaking terms.

In the present article we used the framework of the AKSZ sigma models to develop the field content of the higher gauge theories and, by the material at hand from the previous sections, it was natural to require the structural Lie algebra to be replaced by a general Q-manifold. This is in so far a physical approach as that the notion of Q-manifolds arose in a physical context, namely to describe the supergeometric framework underlying the BV-formulation [37]. In this way, the generalization of the connection 1-forms of a Yang Mills theory to the field content considered sounds much less compulsory than it really is. As mentioned in the introduction, physical considerations often suggest geometrical notions and generalizations into very particular directions. This is also the case in the context of “higher gauge theories”—that is gauge theories where the field content is required to consist locally of a tower of differential forms up to some highest degree p . It is evident that then locally we can introduce (in a canonical way) a graded manifold \mathcal{M}_2 of degree p and interpret the field content as a morphism from $T[1]\Sigma$ to \mathcal{M}_2 . Even without yet

considering gauge transformations acting on these fields, there are sensible requirements on the properties of the theory related to a generalization of the Bianchi identities in ordinary gauge theories that *require* that the target \mathcal{M}_2 has to be a Q-manifold. This approach would have fitted the logic of the present article very well, too, but it would have extended its length further as well so that we decided to not reproduce these considerations, which were presented in [40] and can be found in [17].

Up to now we only considered the generalization of gauge theories in the context of trivial bundles. In principle, it is not at all clear that the local construction generalizes in a straightforward way to a global one. But fortunately here this is indeed the case. We briefly sketch how this works, since, from the *mathematical* point of view, much of the interest in such gauge theories would precisely stem from the global picture. For further details we have to refer the reader to [26].

We again start from ordinary Yang Mills theories. There the space of fields are not Lie algebra valued 1-forms on our spacetime Σ , but rather connections in a principal G -bundle P over Σ . Associated to each principal bundle, however, there is the so-called Atiyah algebroid, which is a very particular Lie algebroid over Σ . As a total space it is equal to TP/G , which is canonically a Lie algebroid by use of the de Rahm differential on TP . The image of its anchor map is $T\Sigma$. Shifting the fiber degrees by 1, this becomes a bundle in the category of Q-manifolds $T[1]P/G \rightarrow T[1]\Sigma$, the typical fiber of which is the Q-manifold $\mathfrak{g}[1]$ (as before, \mathfrak{g} is the Lie algebra of the structure group G). We call this a Q-bundle. Moreover, a connection on P is in bijection to a section of this bundle *viewed* as a bundle of graded manifolds (it is the *flat* connections that are the sections of the Q-bundle).

The generalization is obvious now: As total spaces we consider Q-bundles $\pi: (\mathcal{M}, Q) \rightarrow (\mathcal{M}_1, Q_1)$, where as the base (\mathcal{M}_1, Q_1) we keep $(T[1]\Sigma, d)$ for simplicity, but as typical fibers we permit any (non-negatively graded) Q-manifold (\mathcal{M}_2, Q_2) . Gauge fields are any degree preserving maps $a: \mathcal{M}_1 \rightarrow \mathcal{M}$ such that $\pi \circ a = \text{id}_{\mathcal{M}_1}$. Many constructions that were done for gauge fields $a: \mathcal{M}_1 \rightarrow \mathcal{M}_2$, like the definition of the field strength f , can now be repeated without modification by replacing merely \mathcal{M}_2 by the total space \mathcal{M} . The (generic part of the) gauge symmetries receive the nice interpretation of (possibly a graded subgroup of) vertical inner automorphisms of this bundle. And the gauge invariance of action functionals like (8.12) permits to glue local expressions for functionals together to global ones.

The global setting certainly also opens gates for studying characteristic classes for the bundles. In particular, there exists a generalization of the Chern-Weil formalism. In the classical setting, a connection on a principal bundle together with an ad-invariant polynomial on the Lie algebra gives a cohomology class on the base manifold, which does not depend on the connection and which vanishes when the bundle is trivial. In the context of the classical construction one considers the so-called Weil algebra $W(\mathfrak{g}) = S^*(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ with a certain differential Q_W on it—the connection on P then induces a map from a subcomplex of so-called G -basic elements of W to the de Rahm complex on Σ .

To generalize this construction, one may first observe that W can be identified with differential forms on $\Omega(\mathfrak{g}[1])$, which we can identify with functions on $T[1]\mathfrak{g}[1]$. This reminds one of the target of the map f , namely when $\mathcal{M}_2 = \mathfrak{g}[1]$, cf. the first diagram

above. Given a general Q-manifold \mathcal{M}_2 and a degree preserving map $a: \mathcal{M}_1 \rightarrow \mathcal{M}_2$, can we equip $T[1]\mathcal{M}_2$ with a Q-structure $Q_{T\mathcal{M}_2}$, such that $f: (\mathcal{M}_1, Q_1) \rightarrow (T[1]\mathcal{M}_2, Q_{T\mathcal{M}_2})$ is *always* a Q-morphism? The answer is affirmative [26]:

Proposition 8.3 *The map (6.25) is a Q-morphism, if we equip $T[1]\mathcal{M}_2$ with the Q-structure $Q_{T\mathcal{M}_2} = d + \mathcal{L}_{Q_2}$, where d is the de Rahm differential on \mathcal{M}_2 and \mathcal{L}_v the Lie derivative w.r.t. a vector field v on \mathcal{M}_2 , both viewed as degree 1 vector fields on $T[1]\mathcal{M}_2$:*

$$Q_1 f^* = f^* Q_{T\mathcal{M}_2}. \quad (8.14)$$

Indeed, the canonical lift $Q_{T\mathcal{M}_2}$ of the differential Q_2 to its tangent reproduces the Weil differential Q_W mentioned above for $\mathcal{M}_2 = \mathfrak{g}[1]$. One now generalizes also the notion of being basic, where in the general setting a graded gauge or holonomy group \mathcal{G} plays an important role. In the end one obtains [26] a generalization of the Chern-Weil map in the form

Theorem 8.4 *Let $\pi: \mathcal{M} \rightarrow T[1]\Sigma$ be a Q-bundle with a typical fiber \mathcal{M}_2 , a holonomy group \mathcal{G} , and a “gauge field”, i.e. a section of π (in the graded sense). Then there is a well-defined map in cohomology*

$$H^p(\Omega(\mathcal{M}_2)_{\mathcal{G}}, Q_{T\mathcal{M}_2}) \rightarrow H_{deRahm}^p(\Sigma), \quad (8.15)$$

which does not depend on homotopies of a .

We want to end this section and thus this article by commenting on a rather intriguing link of this construction with the topological sigma models we were considering in a previous section, namely section 6. There is a famous relation of a characteristic class on a principal bundle with a quadratic structure group, the second Chern class or the first Pontryagin class, with the integrand of the Chern-Simons gauge theory: locally “Second Chern form = d (Chern–Simons form)”. This class results from the Chern-Weil formalism by applying the map to the ad-invariant metric κ on \mathfrak{g} , viewed as quadratic polynomial. In sec. 5 we learnt that κ can be viewed as a symplectic form ω of degree 2 on $\mathfrak{g}[1]$. Locally, the above generalized Chern-Weil map is nothing but f^* . On the other hand, the AKSZ sigma model resulted from choosing a QP-manifold $(\mathcal{M}_2, Q_2, \omega)$ as a target—and for the Chern-Simons theory it is this ω that corresponds to κ .

We are thus lead to ask for a relation of the AKSZ sigma models with $f^*\omega$ for a general QP-manifold (that would serve as a typical fiber in a Q-bundle). In Cor. 8.2 we found that the integrand of the AKSZ sigma model can be written as the pullback of the canonical 1-form by f canonical 1-form, if \mathcal{M}_2 is the cotangent bundle $\mathcal{M}_2 = T^*[d-1]\widetilde{\mathcal{M}}_2$ with $\widetilde{\mathcal{M}}_2$ a degree 1 Q-manifold.²⁹ We thus compute $df^*(p_\alpha dq^\alpha) = f^*(d + \mathcal{L}_{Q_2})p_\alpha dq^\alpha$, where we used the chain property (8.14). But the canonical 1-form is invariant w.r.t. the Hamiltonian lift Q_2 of a vector field \widetilde{Q}_2 on its base $\widetilde{\mathcal{M}}_2$; so only the first term remains on the r.h.s., which indeed has the form $f^*\omega$. In fact, this statement that we proved for $\mathcal{M}_2 = T^*[d-1]E[1]$, is true for any QP-manifold; one has [26]

²⁹In fact, this is true for a general Q-manifold, as one can prove directly in generalization of Cor. 8.2 or deduce from Theorem 8.5 below.

Theorem 8.5 *Consider a PQ-manifold \mathcal{M}_2 with symplectic form ω of positive degree $d-1$ and a $(d+1)$ -dimensional manifold N with boundary $\partial N = \Sigma$, and $a \in \text{Mor}(T[1]N, \mathcal{M}_2)$. Then*

$$\int_N f^* \omega = S_{AKSZ}[a], \quad (8.16)$$

the AKSZ sigma model (6.22) on Σ .

As mentioned, these observations generalize the relation of the Pontryagin class to the topological Chern-Simons theory to a much wider range of bundles and topological models.

Certainly, in some cases Q-manifolds and Q-bundles may be “integrable” (already not any Lie algebroid is integrable to a Lie groupoid—cf. [11] for the necessary and sufficient conditions for the integrability of Lie algebroids). Similar to the relation of the Atiyah algebroids to principal bundles, integrable Q-bundles will correspond to rather intricate other type of bundles—in the case of the Lie algebroid Yang-Mills theories, e.g., these are bundles with typical fiber being a groupoid, in the case of the fibers being degree 2 Q-manifolds over a point these will be the nonabelian bundle gerbes of [6]. What we called gauge fields and field strengths and others might call “connections” and “curvatures”, respectively, are often rather intricate to describe in the integrated pictures—in contrast to what happens in the Q-bundle setting.

It is our impression that the Q-bundle picture has several advantages in the description of higher gauge theories. Also, they exist and give rise to potentially interesting action functionals even if the underlying Q-manifolds are not integrable; a topological example of this kind is the Poisson sigma model, which exists and has its merits even if the target Poisson manifold M (the Lie algebroid T^*M) is not integrable.

In any case, we hope to have convinced the reader who was not yet familiar with all the concepts used in this article, that algebroids of various kinds are interesting as mathematical objects on the one hand, interpolating in a novel way between various established notions like tensor fields on manifolds and Lie algebras or symplectic and complex geometry, and, on the other hand, that they appear in many different facets in the context of sigma models and higher gauge theories, in fact in such a way that they have to be considered indispensable for an elegant and technically and conceptually convincing discussion of such theories.

Acknowledgement

We are deeply indebted to V. Cortes for his incredible patience with us in finishing this contribution.

References

- [1] M. Alexandrov, M. Kontsevich, A. Schwartz, and O. Zaboronsky. The geometry of the master equation and topological quantum field theory. *Int. J. Mod. Phys.*, A12:1405–1430, 1997.
- [2] Anton Alekseev and Thomas Strobl. Current algebras and differential geometry. *JHEP*. 0503 (2005), 035.
- [3] Glenn Barnich, Friedemann Brandt, Marc Henneaux. Local BRST cohomology in the antifield formalism. I. General theorems. *Comm. Math. Phys.* Volume 174, Number 1 (1995), 57-91. Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory. *Comm. Math. Phys.* Volume 174, Number 1 (1995), 93-116.
- [4] Martin Bojowald, Alexei Kotov and Thomas Strobl. Lie algebroid morphisms, Poisson Sigma Models, and off-shell closed gauge symmetries. *J.Geom.Phys.* 54 (2005) 400–426.
- [5] Francesco Bonechi and Maxim Zabzine. Lie algebroids, Lie groupoids, and TFT. *Journal of Geometry and Physics* 57:731-744,2007.
- [6] Lawrence Breen, William Messing. Differential geometry of gerbes. *Adv. in Math.* Volume 198, Issue 2, 2005, 732–846.
- [7] Alberto S. Cattaneo, Giovanni Felder. On the AKSZ formulation of the Poisson sigma model. *Lett. Math. Phys.* 56 (2001) 163-179.
- [8] Alberto S. Cattaneo, Jian Qiu, Maxim Zabzine. 2D and 3D topological field theories for generalized complex geometry. arXiv: hep-th/0911.0993.
- [9] Z. Chen, Z. Liu and Y. Sheng. E-Courant Algebroids, International Mathematics Research Notices, doi:10.1093/imrn/rnq053.
- [10] T. J. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.* 319:631–661, 1990.
- [11] Marius Crainic, Rui L. Fernandes. Integrability of Lie brackets. *Ann. of Math.* (2), Vol. 157 (2003), no. 2, 575–620.
- [12] M. Crainic. Generalized complex structures and Lie brackets. arXiv:math/0412097.
- [13] P. Deligne, J. Morgan. Notes on supersymmetry. *Quantum Fields and Strings : A Course for Mathematicians*, vol. 1, AMS (1999).
- [14] S. J. Gates, C. M. Hull and M. Roček. Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models. *Nucl. Phys. B* 248 (1984) 157–186.
- [15] K. Gawedzki and A. Kupiainen. Coset construction from functional integrals. *Nucl. Phys.*, B320:625, 1989.

- [16] J. Grabowski. Courant-Nijenhuis tensors and generalized geometries. *Monografias de la Real Academia de Ciencias de Zaragoza*. 29 (2006), 101–112.
- [17] M. Grützmann, T. Strobl. General Yang-Mills type gauge theories for p-form gauge fields: A physics approach., to appear.
- [18] Markus Hansen, Thomas Strobl. First Class Constrained Systems and Twisting of Courant Algebroids by a Closed 4-form. 25 pages, invited contribution to the Wolfgang Kummer memorial volume, arXiv:0904.0711, 2009.
- [19] Marc Henneaux, Claudio Teitelboim. Quantization of Gauge Systems. Princeton University Press, 552 pp., 1992.
- [20] Nigel Hitchin. Generalized Calabi-Yau manifolds. *Quart. J.Math. Oxford Ser.* 54 (2003) 281–308.
- [21] Noriaki Ikeda. Two-dimensional gravity and nonlinear gauge theory. *Ann. Phys.*, 235:435–464, 1994.
- [22] Noriaki Ikeda. Chern-Simons Gauge Theory coupled with BF Theory. *Int. J. Mod. Phys.* A18 (2003), 2689–2702.
- [23] Ctirad Klimcik and Thomas Strobl. WZW-Poisson manifolds. *J. Geom. Phys.*, 43:341–344, 2002.
- [24] Y. Kosmann-Schwarzbach. Derived brackets. *Lett. Math. Phys.*, 69: 61–87, 2004.
- [25] A. Kotov, P. Schaller, T. Strobl. Dirac Sigma Models. *Commun. Math. Phys.*, 260(2):455–480, 2005.
- [26] Alexei Kotov and Thomas Strobl. Characteristic classes associated to Q-bundles. arXiv:0711.4106, 2007.
- [27] Ulf Lindstrom, Ruben Minasian, Alessandro Tomasiello, Maxim Zabzine. Generalized complex manifolds and supersymmetry. hep-th/0405085.
- [28] Z.-J. Liu, A. Weinstein and P. Xu. Manin triples for Lie bialgebroids. *J. Diff. Geom.*, 45:547–574, 1997.
- [29] Christoph Mayer and Thomas Strobl. Lie Algebroid Yang Mills with Matter Fields. *J. Geom. Phys.* 59:1613–1623, 2009.
- [30] Jae-Suk Park. Topological open p-branes. *Symplectic geometry and mirror symmetry (Seoul, 2000)*, 311–384, World Sci. Publishing, River Edge, NJ, 2001. hep-th/0012141.
- [31] D. Roytenberg. AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories. *Lett. Math. Phys.* 79: 143–159, 2007.

- [32] D. Roytenberg. On the structure of graded symplectic supermanifolds and Courant algebroids. *Quantization, Poisson Brackets and Beyond, Theodore Voronov (ed.), Contemp. Math.*, Vol. 315, Amer. Math. Soc., Providence, RI, 2002.
- [33] Pavol Ševera. Letters to A. Weinstein, unpublished (1998), <http://sophia.dtp.fmph.uniba.sk/~severa/letters/>.
- [34] P. Ševera, A. Weinstein. Poisson geometry with a 3-form background. *Prog.Theor.Phys.Suppl.* 144 (2001) 145–154.
- [35] P. Schaller and T. Strobl. Quantization of field theories generalizing gravity Yang-Mills systems on the cylinder. *Lect. Notes Phys.* 436:98-122, 1994. gr-qc/9406027.
- [36] Peter Schaller and Thomas Strobl. Poisson structure induced (topological) field theories. *Mod. Phys. Lett.*, A9:3129–3136, 1994.
- [37] A. Schwarz. Semiclassical approximation in Batalin-Vilkovisky formalism. *Commun. Math. Phys.*, 158 (2): 373–396, 1993.
- [38] Thomas Strobl. Gravity from Lie Algebroid morphisms *Commun. Math. Phys.*, 246 (3): 475–502, 2004.
- [39] Thomas Strobl. Algebroid Yang-Mills theories. *Phys. Rev. Lett* 93, 211601 (2004).
- [40] Thomas Strobl. Talks given e.g. at Vietri or the IHES in 2005.
- [41] Edward Witten. Nonabelian bosonization in two dimensions. *Commun. Math. Phys.*, 92:455–472, 1984.
- [42] Maxim Zabzine. Lectures on Generalized Complex Geometry and Supersymmetry. *Archivum mathematicum (supplement)*, 42:119–146, 2006.